Transactions Briefs

Cross-Term Analysis of LNL Models

Ho-En Liao and William A. Sethares

Abstract—**This brief proposes a method called cross-term analysis to analyze the structure of the class of LNL models, those which consist of a memoryless nonlinearity sandwiched between two linear systems. By evaluating a family of conditional expectations, cross-term analysis unambiguously determines the cross relations among all delay elements. A byproduct of this methodology is a way to determine the lengths of the two linear subsystems in the LNL model. The method is independent of the degree of nonlinearity of the static subsystem.**

I. Introduction

Identification of block-oriented models and cascade models [3] has been widely discussed due to their special and simple formulations, and these models are useful in a variety of fields. Volterra kernel estimations [4], [5] and correlation analyses [1], [2], [6], [12] are the main tools used to identify these two structures. Nonparametric approaches for Hammerstein and Wiener models are proposed in [9]–[11] where the nonlinear characteristics of the models are estimated from input and output data.

In this brief, a method of cross-term analysis is proposed to identify the structure of the class of LNL models (L denotes FIR filters and N denotes a static nonlinear function) directly from the input/output data. The cross-term analysis of this partitioned system evaluates conditional expectations from a set of input/output data and determines the cross-term set to each delay element as well as the lengths of the linear filters. In direct consequence, classification of LN (known as Wiener systems), NL (known as Hammerstein systems) and LNL models can be made. Once this structural determination has been made, any of the standard methods mentioned above can be used to estimate the appropriate parameters.

In Section II, some structural properties of the class of LNL models are discussed. The cross-term analysis based on system partitions is developed in Section III. In Section IV simulations are conducted to verify the theoretical arguments. Finally, in Section V, conclusions are made.

II. Structural Properties of the Class of LNL Models

The class of LNL models consists of two linear dynamic parts sandwiching a static (memoryless) nonlinear mapping. This cascaded form is shown in Fig. 1. The linear parts of the LNL model are assumed to be FIR filters with lengths L_1 and L_2 respectively, and the system is denoted L_1NL_2. The nonlinear mapping $f(\cdot)$ is assumed to be absolutely square integrable in every closed interval $[a, b]$ on the real line, i.e., $\int_a^b |f(x)|^2\, dx < \infty$, so that $f(\cdot)$ can be expressed

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H.-E. Liao is with the Department of Electrical Engineering, Feng Chia University, Taichung, Taiwan.

W. A. Sethares is with the Department of Electrical and Computer Engineering, University of Wisconsin, Madison, WI 53706 USA.

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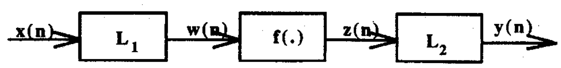

Fig. 1. The L_1NL_2 cascaded model.

via a polynomial expansion. Using the notation of Fig. 1, we have

$$w(n) = \sum_{i=0}^{L_1-1} a_i x(n-i),$$

$$z(n) = f(w(n)) = \sum_{i=0}^{M} \alpha_i w^i(n)$$

and

$$y(n) = \sum_{i=0}^{L_2-1} b_i z(n-i)$$

where M may be infinite. The relation between $x(n)$ and $y(n)$ is given by

$$y(n) = \sum_{q=0}^{M} \left\{ \sum_{k_1}^{L_1-1} \cdots \sum_{k_q=0}^{L_1-1} \alpha_q \sum_{i=0}^{L_2-1} b_i \prod_{j=1}^{q} a_{k_j} x(n-k_j-i) \right\}. \qquad (1)$$

It can be seen from (1) that the structure of the L_1NL_2 model is composed of sums of the products of delay elements. The main purpose of this paper to determine the cross-terms to any given delay element in the model. In addition, the lengths of the two linear systems L_1 and L_2 can be determined.

The L_1N and NL_2 are subclasses of the L_1NL_2 model where $b_0 = 1$ and $b_i = 0$ for all $i \neq 0$, and $a_0 = 1$ and $a_i = 0$ for all $i \neq 0$ in (1), respectively. In addition, the class of LNL models is obviously a subclass of Volterra series expansions [7]. The following remarks come directly from the definition.

R2-1: In the class of LNL models, $LNL \supset LN, LNL \supset NL$ and $LN \cap NL = \{\phi\}$.

R2-2: Let $x(n-k_i)$ be a delay element in the L_1NL_2 model and $x(n-k_j)$ be the cross-term with maximum time lags to $x(n-k_i)$. Then, for fixed i,

$$|k_i - k_j| \equiv \tilde{D} = L_1 - 1.$$

R2-3: Given a L_1NL_2 model for which $L_1 + L_2$ is given, i.e., the delay elements in the model can be predetermined. Then

1) if $\tilde{D} = 0$, then the system is NL_2;
2) if $\tilde{D} = L_1 + L_2 - 2$, then the system is L_1N.

III. Identification of Cross-Terms

This section proposes a method to identify the cross-term set of a delay element in a L_1NL_2 model. This information can then be combined with the results of the previous section to determine which of the models, NL, LN, or LNL is most appropriate to describe the given I/O characteristic. The basis for the cross-terms determination is a partitioned representation of the nonlinear system, which is described first.

A Volterra series expansion of arbitrary order can be partitioned with respect to the ith delay element as

$$y(n) = \sum_{k=1}^{M_i} f_k(x(n-i)) g_{i,k}(C_{i,k}) + v_i(\overline{C}_i) \qquad (2)$$

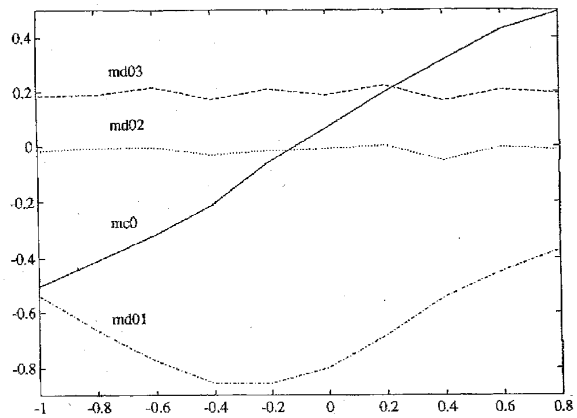

Fig. 2. The plot of conditional expectations corresponding to $i = 0$.

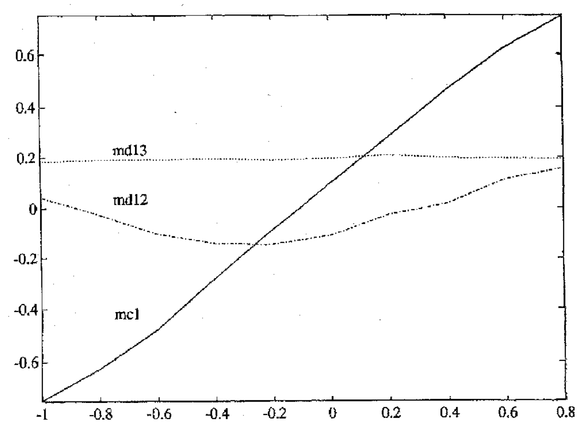

Fig. 3. The plot of conditional expectations corresponding to $i = 1$.

where $C_{i,1} \cup C_{i,2} \cup \cdots \cup C_{i,M_i} \equiv \tilde{C}_i$ is the cross-term set of $x(n-i)$. $\overline{C}_i$ denotes the set of delay elements that does not contain the delay element $x(n-i)$. The upper bound of the summation M_i is determined by the structure of the model and $x(n-i)$. Hence, the first term in the right-hand side of (2) completely specifies the nonlinear contribution of the delay element $x(n-i)$ to the I/O characteristic of the system and the second term is uninfluenced by $x(n-i)$.

The establishment of the partitioned form is rather intuitive and straightforward. For example, if a mathematical description of a system is

$$y(n) = ax(n)x^2(n-3)x(n-4) + bx^2(n-2)x^3(n-3) + cx(n-5)x(n-7) + dx^2(n)$$

and the delay element $x(n)$ is of interest, then the partitioned representation of the above system with respect to $x(n)$ can be with $M_0 = 2, f_1(x(n)) = ax(n), f_2(x(n)) = dx^2(n), C_{0,1} = \{x(n-3), x(n-4)\}$ (therefore, $f_{0,1}$ is a function of $x(n-3)$ and $x(n-4)$ and is equal to $x^2(n-3)x(n-4)), C_{0,2} = \{\phi\}$ (therefore, $f_{0,1}$ is the identity mapping), and $\overline{C} = \{x(n-3), x(n-2), x(n-5), x(n-7)\}$ (therefore $v_0(\overline{C}_0) = bx^2(n-2)x^3(n-3) + cx(n-5)x(n-7)$). Note that $(C_{i,1}C_{i,2}) \cap \overline{C}_0 = \{x(n-3)\}$.

After introducing the partitioned form based on the Volterra series expansions, we are now well equipped to determine the cross-term set $\tilde{C}_i$ of $x(n-i)$.

Lemma: Let $\boldsymbol{x}$ be i.i.d. and bounded in the interval $[c, d]$ and $f(\cdot)$ be a continuous function of $\boldsymbol{x}$ (not a constant) in a closed interval $[p, q] \supset [c, d]$. Then there exists a closed interval $I \in [c, d]$ such that

$$E[f(\boldsymbol{x})] \neq E[f(\boldsymbol{x}) | \boldsymbol{x} \in I].$$

Proof of the above lemma follows from the mean-value theorem [8].

Proposition:

Consider the partitioned form of the system in (2) with i.i.d. input sequence $\boldsymbol{x}$. Let I_a be an arbitrary closed interval. Then there exists a fixed closed interval I_b such that the value of $E[y(n)|x(n-i) \in I_a] - E[y(n)|x(n-i)I_a, x(n-j)I_b]$ is independent of I_a iff $x(n-j)$ is not a cross-term to $x(n-i)$.

Proof: Note that

$$E[y(n)|x(n-i) \in I_a] = \sum_k E[f_k(x(n-i)|x(n-i) \in I_a] \cdot E[g_{i,k}(C_{i,k})] + E[v_i(\overline{C}_i)] \quad (3)$$

because the input sequence is i.i.d. Moreover

$$E[y(n)|x(n-i) \in I_a, x(n-j) \in I_b] = \sum_k E[f_k(x(n-i))|x(n-i) \in I_a] \cdot E[g_{i,k}(C_{i,k})] + E[v_i(\overline{C}_i)|x(n-j) \in I_b] \quad (4)$$

when $x(n-j)$ is not a cross-term to $x(n-i)$ and

$$= \sum_k E[f_k(x(n-i))|x(n-i) \in I_a] \cdot E[g_{i,k}(C_{i,k})|x(n-j) \in I_b] + E[v_i(\overline{C}_i)|x(n-j) \in I_b] \quad (5)$$

when $x(n-j)$ is a cross-term to $x(n-i)$. Whenever $x(n-j)$ is not a cross-term, then (3)–(4) gives

$$E[v_i(\overline{C}_i)] - E[v_i(\overline{C}_i))|x(n-j) \in I_b] \equiv h(I_b)$$

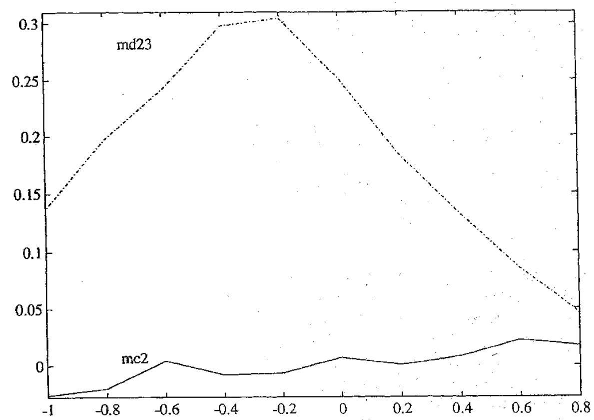

Fig. 4. The plot of conditional expectations corresponding to $i = 2$.

TABLE I
THE STANDARD DEVIATIONS (std) OF $md_{i,j}$'S

std (i \ j)	1	2	3
0	0.1727	0.0160	0.0178
1	—	0.1049	0.0064
2	—	—	0.0965

and $h(I_b)$ is independent of the interval I_a chosen. This proves the necessary condition.

For sufficiency, suppose $x(n-j)$ is a cross-term to $x(n-i)$. Then (3)–(5) gives

$$\sum_k E[f_k(x(n-i))|x(n-i)\in I_a] \cdot \{E[g_{i,k}(C_{i,k})] - E[g_{i,k}(C_{i,k})|x(n-j)\in I_b]\} + E[v_i(\overline{C}_i)] - E[v_i(\overline{C}_i))|x(n-j)\in I_b] \equiv h'(I_a, I_b)$$

The term in the braces of the above equation can be nonzero with the appropriate choice of I_b by the Lemma. Hence, when different intervals are taken (those conditioned on $x(n-i)$), $h'(I_a, I_b)$ is not constant, but depends on the interval I_a chosen.

The proposition above suggests a way to identify cross-term sets by comparing the conditional expectations of different intervals provided that the input data is i.i.d. Moreover the cross-term analysis as suggested by the Proposition is not constrained to LNL model but is applicable to very general nonlinear systems whenever they are representable using Volterra series expansions.

IV. Simulations

The following simulation shows an example of the determination of cross-term sets by evaluating conditional expectations under the assumption that $L_1 + L_2$ is predetermined.

Consider the "unknown" L_1NL_2 system with

$$\begin{aligned} L_1:&\quad w(n) = 0.5x(n) + 0.45x(n-1);\\ N:&\quad x(n) = -0.6, \text{ when } w(n) \le 0.3\\ &\quad\quad = 2w(n), \text{ when } -0.3 < w(n) < 0.3\\ &\quad\quad = 0.6, \text{ when } w(n) \ge 0.3;\\ L_2:&\quad y(n) = z(n) + 0.62z(n-1) - 0.5z(n-2). \end{aligned}$$

The conditional expectations are evaluated by averaging ten independent experiments. In each experiment, the input data (with length of 5000) are i.i.d. and uniformly distributed in [-1, 1]. The intervals I_a's are chosen to be equally spaced in [-1, 1] with length 0.2 and $I_b = [0.5, 1]$. Let mc_i denote $E[y(n)|x(n-i)\in I_a]$, and let $md_{i,j}$ denote the difference between (3) and (4) or (5). The results are shown in Figs. 2, 3, 4 with i = 0, 1, 2, respectively. The standard deviations of $md_{i,j}$ are also shown in Table I, from which we can see that the standard deviations of $md_{0,2}, md_{0,3}, md_{1,3}$ are all small, while the others are large. Thus, the cross-term sets are

$$\begin{aligned} \tilde{C}_0 &= \{x(n), x(n-1)\}\\ \tilde{C}_1 &= \{x(n), x(n-1), x(n-2)\}\\ \tilde{C}_2 &= \{x(n-1), x(n-2), x(n-3)\} \end{aligned}$$

and

$$\tilde{C}_3 = \{x(n-2), x(n-3)\}.$$

Since the longest- (or shortest-) delay cross-term to any of the delay element $x(n-i)$ is $x(n-i-1)$ (or $x(n-i+1)$), $\hat{D} = L_1 - 1 = 1$ by R2-2 and $L_2 = 3$ since the longest-delay element in the model is $x(n-3)$, i.e., $L_1 + L_2 - 2 = 3$.

V. Conclusion

This brief represents the structural properties of the class of L_1NL_2 models and a way to find the cross-term sets within the models. The lengths of the linear filters can be also determined. The proposed cross-term analysis can be used to predetermine the structure of an unknown model so that the parameter estimation of the unknown model is facilitated.

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An Inequality Concerning Nonexpansive Mappings

Irwin W. Sandberg

***Abstract*—In recent results concerning nonexpansive maps and the problem of iteratively solving nonlinear equations of the form $Qx = y$ in a Hilbert space (application areas include networks and signal processing), a certain inequality plays a central role. Here we consider the case in which Q is continuously Fréchet differentiable and we give criteria under which the inequality is satisfied.**

I. Introduction

The problem of iteratively solving equations of the form

$$Qx = y \tag{1}$$

for a solution x, given y and an operator Q, arises in several contexts in the Circuits and Systems area. A condition that plays a central role in several recent results is that Q takes a (real or complex) Hilbert

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The author is with the Department of Electrical and Computer Engineering, The University of Texas at Austin, Austin, TX 78712 USA.

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