Tuning, Tonality, and Twenty-Two-Tone Temperament

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Introduction

Western modal and tonal music generally conforms to what is known as the 5-limit; this is often explained (e. g., by Hindemith) by stating that the entire tuning system is an expression of integer frequency ratios using prime factors no higher than 5. A more correct description is that all 5-limit intervals, or ratios involving numbers no higher than 5, aside from factors of 2 arising from inversion and extension, are treated as consonances in this music, and thus need to be tuned with some degree of accuracy. The diatonic scale of seven notes per octave has been in use since ancient Greek times, when harmonic simultaneities (other than unisons and octaves) were not used and the only important property of the scale was its melodic structure. As melodic considerations continued to play a dominant role in Western musical history, scales other than the diatonic had very limited application, and harmonic usage was intimately entangled with diatonic scale structure. It is fortuitous that the diatonic scale allowed for many harmonies that displayed the psychoacoustic phenomenon of consonance, defined by small-integer frequency ratios, as well as many that did not.

Medieval music conformed to the 3-limit, the only recognized consonances being the octave (2:1) and the perfect fifth (3:2) (plus, of course, their octave inversions and extensions, obtained by dividing or multiplying the ratios by powers of 2; this "octave equivalence" will be implicit from here on.[1]) The diatonic scales were therefore tuned as a chain of consecutive 3:2s (Pythagorean tuning). By the end of the 15th century,[2] the 5-limit had eclipsed the 3-limit as the standard of consonance and two new consonant intervals were recognized: the major third (5:4) and the minor third (6:5). It was now possible to create consonant triads, consisting of all three non-equivalent consonant intervals and no dissonant intervals. There are two such triads: major (6:5:4), and minor (1/4:1/5:1/6).[3] The advent of the 5-limit brought with it a new temperament of the diatonic scale, the meantone. Typical varieties of meantone are well approximated in 19- and 31-tone equal temperament; 12-tone equal temperament is approximately 1/11-comma meantone,[4] lying about halfway between typical meantone temperaments and Pythagorean tuning.

Soon anything less than a triad was deemed an incomplete harmony. Use was also made of linked 5-limit (or stacked-thirds) chords ("major and minor seventh chords") and linked 3-limit chords ("suspended chords"), which contained only one dissonant interval but had to resolve. More dissonant triads ("diminished") and tetrads ("dominant, diminished, and half-diminished seventh chords"), formed from the same pattern of scale steps as the 5-limit sounds, were normally restricted to cadential or

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[1] The fact that the phenomenon of octave similarity does not lead to exact equivalence of harmonic function is assumed to have a minor impact on the general conclusions reached by this paper.

[2] Fokker, Adriaan D. 1975. *New Music with 31 Notes.* Verlag für systematische Musikwissenschaft GmbH, Bonn, p. 35.

[3] Note that the 6:4 interval in these chords is equal to 3:2, the perfect fifth. Recall also that we are implicitly defining all octave inversions and extensions of these intervals and chords as consonant. The fact that certain styles treat 4:3 and its extensions as a dissonance when it includes the bass part is a detail that need not concern us here.

[4] Barbour, J. Murray. 1951. *Tuning and Temperament: A Historical Survey.* Michigan State College Press, East Lansing, p. 35.

modulatory progressions. These chords resulted from the existence of a diminished fifth within the diatonic scale. Unlike most fifths in the scale, the diminished fifth was considered dissonant, since it did not approximate any 5-limit consonances. A complete resolution was not felt until both members of the diminished fifth had moved to a consonant chord. The major and minor modes gained supremacy because only in those modes was the diminished fifth disjoint from the tonic triad.

As early as the mid-eighteenth century, Tartini observed that the "harmonic 7th is not dissonant but consonant. . . . it has no need either of preparation or resolution: it may equally well ascend or descend, provided that its intonation be true."[5] So it seems natural to try to expand harmony by making 7-limit intervals the fundamental harmonic units. The new consonant intervals would be 7:4, 7:5, and 7:6. There would be a major tetrad (7:6:5:4) and a minor tetrad (1/4:1/5:1/6:1/7), each consisting of the six non-equivalent 7-limit intervals. However, these chords occur in only one position and with questionable accuracy in the diatonic system, relatively pure versions obtainable only by certain chromatically altered chords (such as the augmented sixth) when played in 31-tone equal temperament or similar meantone systems. Therefore an entirely new scale is required, in which the tetrads can act as the basic consonances.

Equal temperaments

Figure 1 (all figures are below the text portion of this document) shows the adequacy with which the simplest equal-tempered tunings approximate all justly tuned intervals of the 3- through 15-limits.[6] The value assigned to the "accuracy" of each interval is proportional to an estimate of the likelihood of the nearest equal-tempered approximation being heard as the just interval it represents. A normal probability distribution is assumed, centered on the actual interval with an accuracy value of 1.0 assigned to the peak, and with a standard deviation of 1% in log-frequency space.[7] We do not go beyond 34 tones because this would allow two notes of the tuning to be within 1% of a given target pitch. The accuracy of the tuning as a whole is defined as the geometric mean of the accuracies of the intervals considered so that (a) even if only one interval is badly out-of-tune, the entire tuning receives a poor rating; and (b) the rank-order of the results for a given limit is just that of the mean squared errors and is independent of the standard deviation assumption.[8]

In non-harmonic music, the purity of consonances is considerably less important, and intervals beyond the 3-limit are of negligible importance. Figure 1 suggests that among equal temperaments, scales of 5 and 7 notes per octave ought to be candidates for tuning melodic music, and one indeed finds many approximations to these scales throughout the world. Thai music, which requires equal temperament so that it can modulate, is played in 7-tone equal temperament, and the 5-tone equal

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[5] Partch, Harry. 1974. *Genesis of a Music.* Da Capo Press, New York, p. 388.

[6] For the limit λ, all ratios a/b of odd numbers a and b such that $1 \leq b < a \leq \lambda$ are evaluated. 9:3 is considered a separate, 9-limit interval, since it occurs in addition to the 3:1 in complete chords of the 9-limit or higher.

[7] Terhardt sees consonance judgments as having two components, complex pitch formation and roughness. Terhardt, E. 1974. "Pitch, consonance and harmony." *J. Acoust. Soc. Amer.* Vol. 55 p. 1061. According to Goldstein, the precision with which frequency information is transmitted to the brain's central pitch processor is between 0.6% and 1.2% within a certain optimal frequency range. Goldstein, J. L. 1973. "An optimum processor theory for the central formation of the pitch of complex tones." *J. Acoust. Soc. Amer.* Vol. 54 p. 1499. The roughness component is somewhat more permissive as to mistuning, but a 1% standard deviation would cause the accuracy to fall below ½ at around the point where benign beating begins to gives way to the roughness effects of the critical band.

[8] If the tuning is not *consistent* within a given limit -- that is, if for some odd numbers a, b, and c less than or equal to the limit, the best approximation of b:a plus the best approximation of c:b does not equal the best approximation of c:a -- then no accuracy value is computed. Consistency beyond the 15-limit is not achieved with fewer than 58 equal steps per octave.

tempered scale is widespread in Africa. The huge gain in the accuracy of 3-limit intervals obtained by using a 12-tone scale (or approximation thereof) is what allowed Chinese and medieval Western music to develop dyadic harmony. It turned out that 12-tone equal-tempered scales, and the diatonic modes that were in use in the West, were conducive to 5-limit (triadic) harmony as well, with a decent degree of accuracy (about 78% of perfection, according to figure 1).

Although the move to 19- or 31-tone equal temperament has been proposed time and again for the greater consonance that these tunings would afford to modal and tonal triadic music, it has been felt that the potential aesthetic advantages are greatly outweighed by the practical disadvantages, which could include the need to invent and learn to play a new set of (more complex) instruments. These sorts of difficulties have been surmounted by Partch, Ben Johnston, and the Dutch 31-toners,[9] and advances in electronic music have made such explorations more widespread. In any case, if we want a tuning that is at least as accurate in the 7-limit as 12-equal is accurate in the 5-limit, fig. 1 tells us to consider 22, 26, 27, and 31-tone tunings. It will turn out that only 22-equal contains a system analogous to the diatonic system, but which involves harmonies of the 7-limit.

It may be objected that higher-limit intervals may need to be tuned more accurately than lower-limit ones, since they are more apt to be confused with other intervals. If the intervals are often sounded alone, this is indeed the case. Partch stated in his “Observation One” that the “field of attraction”[10] of an interval is inversely proportional to its “Identity,” or limit. A similar result can be derived from the work of van Eck, when errors in his reasoning are corrected.[11] Figure 2 is a modification of fig. 1 in which the standard deviation used in evaluating each interval is inversely proportional to its limit.[12] The standard deviation used for 5-limit intervals is still 1% (this again has no effect on the rank-order of the results within a given limit). The same four tunings appear best for the 7-limit, but only two are still more accurate than 5-limit 12-equal: 27 and 31. However, we are interested in a style where complete harmonies, not isolated intervals, are the norm, and in complete harmonies the context of simpler intervals seems to aid in the recognition of the more complex ones. For example, the tritone b-f in 12-equal is ambiguous when heard in isolation, but can be heard as 7:5, 10:7, or 17:12 in the chords g-b-d-f, c#-g#-b-f, and e-g#-d-b-f, respectively.[13] Additionally, the argument can be made that although the recognizability of simpler ratios is reduced more slowly than that of more complex ratios when both are detuned by the same amount, the consonance of the simple ratios deteriorates at least as rapidly. Therefore, we will consider figure 1 to be more important for our purposes.

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[9]Fokker., op. cit.

[10]Partch, p. 184

[11]The errors involve viewing pitch height here as a linear, and there as a logarithmic, function of frequency. Using a logarithmic (e.g., cents) scale consistently would lead to the result that the size of an acceptable range for any just ratio is inversely proportional to the denominator of the ratio in lowest terms (the numerator is assumed to be greater than the denominator). See the 1994 version of this paper (unpublished) for a derivation of this result. Since the denominator of the ratio is never larger than the (odd-number) limit, the greatest lower bound on the sizes of the acceptable ranges for an interval and all its octave equivalents is inversely proportional to the limit. van Eck, C. L. Van Panthaleon. 1981. *J. S. Bach’s Critique of Pure Music.* Princo, Culemborg, the Netherlands, Appendix II. For a justification for the definition of these acceptable ranges, see Mann, Chester D. 1990. *Analytic Study of Harmonic Intervals.* Tustin, Calif.

[12]Defined as the larger of the two odd numbers defining the interval.

[13]This feature is important in jazz harmony under the rubric of “tritone substitution.”

Generalizing diatonicity

Many of the previous authors who have tried to generalize the diatonic scale have ignored the fact that 12-equal is not the historical basis for the diatonic scale; in fact the converse is true, and 12-equal was chosen over other meantone tunings only for convenience. Balzano focuses on certain group-theoretical properties of 12-equal and of the diatonic scales within it. For example, his view of harmonic structure hinges on the fact that in 12-equal, the size of the minor third (3) times the size of the major third (4) equals the size of the entire set. He proceeds by finding a nine-tone scale in 20-equal with many of the same properties.[14] Had he taken the diatonic scale from 19- or 31-equal, systems that musicians actually considered before 12-equal won out, he would not have found many of his properties. Likewise, Clough and Douthett's maximal evenness property[15] fails if the diatonic scale is taken from any tuning other than 12-, 19-, or 26-equal. Yasser also assumes a 12-tone tuning in defining the diatonic system as "7+5=12" (seven diatonic plus five altered notes make the 12 chromatic notes), which allows him to posit the evolutionary series "2+3=5, 5+2=7, 7+5=12, 12+7=19, 19+12=31" where the chromatic notes in one system become the diatonic notes in the next system, and the diatonic notes in one system become the altered notes in the next system.[16]

Many other theorists, including Schoenberg, have explained the diatonic scale as arising from three consecutive triads in a chain of fifths. This is another reversal of historical facts, as the chords were constructed from the scale, and not the other way around. Von Hoerner constructed a scale from three consecutive 7-limit tetrads in a chain of fifths, using 31-tone equal temperament.[17] The structure of this scale is too bizarre for it to function as a melodic entity.

A few other theorists have attempted to preserve some important features of the diatonic system while proposing musical tones with unusual harmonic spectra. Bohlen and Pierce independently discovered that the equal-tempered division of the 3:1 (perfect twelfth) into 13 parts contains a nine-tone scale with excellent 9:7:5:3 and 1/3:1/5:1/7:1/9 chords.[18] Pierce suggests that using timbres without even-numbered harmonic partials would not only highlight the consonance of these chords but would also cause the phenomenon of octave equivalence to give way to one of "tritave equivalence," or equivalence at the perfect twelfth. However octave equivalence seems pervasive, regardless of the overtone structure of the timbres used. Goldsmith proposed using tones with inharmonic overtone series to go along with a division of the octave into 16 equal parts. Proceeding by analogy from the major scale, a nine-tone scale is derived, with three disjoint triads, which could be rendered somewhat consonant with an appropriate overtone structure.[19] However, these triads do not contain anything resembling a 3:2 interval, which is what confers stability (and a root) to the triads of diatonic music. We will restrict our attention to tones with harmonic overtone series because they evoke the sensation of a single pitch and arise from the widest variety of acoustic and electronic methods of tone generation.

The diatonic scale has many properties that allow it to form the melodic basis for a 5-limit harmonic style. A few definitions will allow us to generalize these properties. Given any odd-number harmonic limit, we can define the set of consonant intervals as all ratios of odd numbers less than or

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[14]Balzano, G. J. 1980. "The Group-Theoretic Description of 12-Fold and Microtonal Pitch Systems," *Comp. Music J.* Vol. 4, No. 4, pp.66-84.

[15]Clough, John and Douthett, Jack. 1991. "Maximally Even Sets." *Journal of Music Theory* Vol. 35 p. 93.

[16]Yasser, Joseph. 1932. *A Theory of Evolving Tonality.* American Library of Musicology, New York.

[17]von Hoerner, Sebastian. 1974. "Universal Music", *Psychology of Music* vol. 2, 1974, pp. 18-28.

[18]Mathews, M. V., Pierce, J. R., and Roberts, L. A. 1987. "Harmony and New Scales." In: *Harmony and Tonality.* J. Sundberg, ed. Royal Swedish Academy of Music, Stockholm, pp. 59-84.

[19]Goldsmith, David S. 1971. "An Electronically Generated Complex Microtonal System of Horizontal and Vertical Harmony", *Journal of the Audio Engineering Society* vol.19 no. 10 pp. 851-858.

equal to the limit (as well as their octave equivalents, which will contribute factors of 2 to the numerator or denominator of these fractions), and we define the complete consonant chords as those which contain all of the consonant intervals and no dissonant (non-consonant) intervals. We will use the following symbols for tempered intonations of just intervals and their octave equivalents: Q will represent any tempered intonation of 3:2 (or 4:3, 3:1, etc.); T—5:4 (or 8:5, 5:2, etc.); S—7:4 (or 8:7, 7:2, etc.); and U—1:1 (or 2:1, 4:1, etc.). For example, any meantone system has 4Q=T. The Q, the strongest interval within the chord, determines the root of a consonant chord: the root is that member of the Q that represents the number 2 in the ratio 3:2. Let us define a "characteristic dissonance" as any dissonant interval in the scale that is the same size, as defined by number of scale steps, as a consonant interval. The only example in the unaltered diatonic scale is the diminished fifth (and, of course, the augmented fourth). The properties that we would like to reproduce in our scale are as follows:

(0) Octave equivalence:

There is a basic scale (a subset of the tuning) which repeats itself exactly at the octave, extending infinitely both upwards and downwards in pitch.

Octave similarity is universally perceived, even by some animals.[20]

(1) Scale structure:

(Version a - distributional evenness): The basic scale has two step sizes, and given these step sizes, the notes are arranged in as close as possible an approximation of an equal tuning with only as many notes per octave as the basic scale.[21]

(Version b - tetrachordality): The basic scale has a structure emphasizing similarity at the Q. In particular, there is a "tetrachordal" structure, that is, within any octave span, the pattern of steps within one approximate 4:3 are replicated in another approximate 4:3, with the remaining "leftover" interval spanned using patterns of step sizes (often just one step) found in the "tetrachord."

This scale defines the "key" or "mode"; the set of unaltered pitches used in any section of a composition.[22]

These properties are enough to ensure an intelligible melodic framework. The following properties allow harmony to be based on this framework.

(2) Chord structure:

There exists a pattern of intervals (defined by number of scale steps, not specific as to exact size) which produces a complete, consonant chord on most scale degrees.

This condition provides for a formal rule governing the origin and use of the consonant chords, so that they can become recognized, after a reasonable period of exposure to the system, as the structural harmonies. Or, as Krumhansl puts it,

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[20]Blackwell, H. and Scholsberg, H. 1943. "Octave Generalization, Pitch Discrimination and Loudness Threshold in the White Rat." *Journal of Experimental Psychology* vol. 33 p. 407.

[21]An equivalent condition is that every generic interval (measured in steps of the basic scale) comes in at most two specific sizes. Clough, John, personal communication.

[22]Either of the conditions above provide melodic smoothness and ease of intelligibility to the mode, although I find, where the two conditions yield differing results, that version (b) leads to more pleasing and comprehensible scales. Future musicological research and psychological experiments may determine which version has more validity.

> If chord construction is determined in some principled way by scale structure, then this further serves to maintain the tonal framework for encoding pitch information.[23]

(3) Chord relationships:
The majority of the consonant chords have a root that lies a Q away from the root of another consonant chord.
This ensures the existence of simple chord relationships that could serve as the basis for comprehensible chord progressions and modulations.

(4) Key coherence:
A chord progression of no more than three consonant chords is required to cover the entire scale.
This restriction should suffice to ensure that the sense of "key" or position within the scale is never lost in the course of the music, and to allow a new key to be easily established.

While these properties may suffice for a "modal" style, the rest of the properties have to do with certain special, "tonal" modes of the basic scale.

(5) Tonicity:
The notes of the scale are ordered, increasing in pitch, so that the first note is the root of a complete consonant chord, defined hereafter as the "tonic chord."
This is more a numbering convention than a true property.

There are two ways of formulating the remaining properties. The stronger will define the primary tonal modes, while the weaker will define secondary modes whose tonic chords are points of intermediate stability.

<u>Strong Version</u>

(6) Homophonic stability:
All characteristic dissonances are disjoint from the tonic chord, with the following possible exception: A characteristic dissonance may share a note with the tonic chord if, when played together, they form a consonant chord of the next higher limit ($3\Rightarrow5$, $5\Rightarrow7$, $7\Rightarrow9$).
This ensures that the scale will sound at least as consonant against the tonic chord as against any other chord.

(7) Melodic guidance:
The rarest step sizes are only found adjacent to notes of the tonic chord, acting as "signposts" if not necessarily leading tones *per se*.
As Richmond Browne has pointed out, "Rare intervals aid position finding."[24]

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[23]Krumhansl, Carol L. 1987. "General Properties of Musical Pitch Systems: Some Psychological Considerations." In: *Harmony and Tonality*. J. Sundberg, ed. Royal Swedish Academy of Music, Stockholm, p. 37.
[24]Browne, R. 1981. "Tonal Implications of the Diatonic Set." *In Theory Only* vol. 5 no. 6 pp. 3-12.

Weak Version

(6) Homophonic stability:

At least one characteristic dissonance either is disjoint from the tonic chord, or shares a note with the tonic chord such that, when played together, they form a consonant chord of the next highest limit ($3\Rightarrow5$, $5\Rightarrow7$, $7\Rightarrow9$).

This provides for a "tonicizing" interval, which bears a special structural relationship to the tonic chord. When a characteristic dissonance is disjoint from the tonic, it resolves to it, producing a "dynamic" tonality; when a characteristic dissonance blends with the tonic chord to form a stable quasi-consonant sonority, it produces a "static" tonality.

Let's see how these rules apply to 5-limit and 3-limit harmony. The scales are illustrated by the division of the octave into large (L) and small (s) steps. We will use rules 5-7 as a "sieve" to remove those modes that do not satisfy all of them.

5-limit: The diatonic scale.

(1) The octave is divided into five large and two small steps: 5L+2s=U. Possible tunings: 12- (L=2, s=1), 19- (L=3, s=2), 26- (L=4, s=3), or 31- (L=5, s=3) tone equal temperaments, i.e., any meantone system.

(a) The two small steps are as far apart (in a cyclic sense) as possible.

(b) The two tetrachords are each composed of one small plus two large steps, with an additional large step filling in the octave.

(2) The consonant intervals are Q=3L+s, T=2L, and Q-T=L+s. The pattern 1, 3, 5 produces a consonant triad 6 times out of 7.

(3) The six consonant chords form a chain of Qs.

(4) Any three consecutive chords in the chain will cover the entire scale.

(5) Removes the locrian (s L L s L L L) mode.

Strong Version

(6) Removes the phrygian (s L L L s L L) and lydian (L L L s L L s) modes, since they contain a note a diminished fifth (s+L+L+s, the characteristic dissonance) above the 5th and below the 1st scale degrees, respectively. The mixolydian (L L s L L s L) and dorian (L s L L L s L) modes each contain a note a diminished fifth from the 3rd scale degree, but it forms a rough 7-limit tetrad with the respective tonic triads in 12-equal.

(7) Removes the mixolydian and dorian modes,[25] leaving the usual major (L L s L L L s) and natural minor (L s L L s L L) modes.

Weak Version

(6) Since there is only one characteristic dissonance, the result is the same as for (6) above: major, natural minor (both dynamic), and possibly in 12-equal mixolydian and dorian (both static).

[25]These two modes are commonly used in jazz, though, where the approximate 7-limit tetrads on the root (in accordance with (6)) are often used as "tonics", and (7) can be considered to be satisfied in that the "half steps" are indeed found adjacent to notes of these tetrads. Schenker labeled the mixolydian "Mischung 3" and the dorian "Mischung 5." Schenker, H. 1906. *Harmonielehre.* Union Deutsche Verlagsgesellschaft, Stuttgart. Universal Edition, Wien, 1935.

3-limit: The pentatonic scale.[26]

(1) 2L+3s=U. Possible tunings: 7- (L=2, s=1), 12- (L=3, s=2), 17- (L=4, s=3), 19- (L=5, s=3), 22- (L=5, s=4), 26- (L=7, s=4), 27- (L=6, s=5), 29- (L=7, s=5), 31- (L=8, s=5) equal, or any tuning where the interval L+2s forms a Q of less than 720 cents.

(a) The two large steps are as far apart (in a cyclic sense) as possible.

(b) The two "trichords" are each composed of one small plus one large step, with an additional small step filling in the octave.

(2) The consonant interval is Q=L+2s. The pattern 1, 4 produces a consonant dyad 4 times out of 5.

(3) The four consonant chords form a chain of Qs.

(4) Any three chords that are not all consecutive in the chain will cover the entire scale.

(5) Removes the L s L s s mode.

Strong Version

(6) The characteristic dissonance is s+s. The major (s s L s L) and minor (L s s L s) pentatonic modes contain a note that is this interval above the 1st and below the 4th degrees of the scale, but, especially in a meantone system, this note will form a 5-limit triad with the tonic chord.

(7) Removes the s L s L s and s L s s L modes, leaving the major and minor pentatonic modes, at least in meantone tunings.[27]

Weak Version

(6) The s L s L s and s L s s L modes (both dynamic) both qualify in addition to the major and minor pentatonic modes (both static).

We can even consider a 3-tone scale, a bare "tetrachordal" framework, to be a sort of degenerate 1-limit case:

(1) 2L+s=U. Any tuning where L is a recognizable Q will do, including 5-equal (L=2, s=1).

(a) Is trivially satisfied.

(b) May be questionable because the "leftover" interval has no counterparts within the empty "dichords".

(2) A consonant "monad" exists on every scale degree.

(3) The three notes form a chain of Qs.

(4) The three monads are the notes of the scale.

(5) Says nothing.

Strong Version

(6) The characteristic dissonance is s, removing the s L L and L L s modes.

(7) Removes the L s L mode, leaving none.

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[26]See the discussion of "Infra-Tonality" in Yasser, op. cit.

[27]Daniel Wolf has suggested that in Thai and Cambodian music,

> The fixed pitched instruments may be said to approach a 7-equal tuning, but singers and instruments with variable pitches sing or play distinctly major or minor thirds, depending upon position in the 5-tone scale in use. The basic scalar template is 1 2 3 5 6, the interval 1-3 is almost always "major" and the 3/5, 6/1 intervals are almost always minor. My sense is that the *inner melody* of the music is vocal, but that the fixed pitch instruments provide a framework for transposition.

The most commonly used modes in Thai music are analogous to the familiar major and minor pentatonic modes (one of the secondary modes (s L s L s) is occasionally used as well). Thus (6) may be satisfied fairly well particularly for those timbres with harmonic partials. Wolf, Daniel, Sept. 21, 1996. Alternate tuning mailing list, digest #842, topic #2.

Weak Version

(6) Only the L s L (dynamic) mode is weakly tonal.

With version (a) of rule (1), we can almost add three scales to our list: the augmented (hexatonic) scale (1, 4 is always consonant), the diminished (octatonic) scale (1, 4, 7 is always consonant),[28] both of which fail criterion (3); and Blackwood's ten-tone symmetrical scale in 15-equal,[29] (1, 4, 7 is always consonant) which just misses property (4). The "just" major and minor diatonic scales[30] satisfy properties (2) through (5), but fail (1a) since they have three step sizes, and fail (1b) because not every octave span contains two identical tetrachords.

To the 7-limit

Some identities for the equal tunings we will be considering are as follows:

22	26	27	31
22Q=U	26Q=U	27Q=U	31Q=U
9Q=T	4Q=T	9Q=T	4Q=T
-2Q=S	-9Q=S	-2Q=S	10Q=S
2(S-T)=U	2(S-T)=U	3T=U	3S+Q=U

In searching for a scale with tetradic harmonies, rule (4) limits us to scales of twelve or fewer notes. Rule (3) tells us that the scale will consist of strings of Qs. A single string of Qs, as in the diatonic scale, is not enough to produce sufficient Ts and Ss in any of the tunings above. However, the last row of identities suggests that we might try deriving the notes of the scale from 2 or 3 strings of Qs. Such scales in 31-tone equal temperament cannot satisfy either version of property (1) without exceeding fourteen notes. In 27-equal, a twelve-tone symmetrical scale satisfying (1b) exists, but it fails condition (2) miserably.

Only if we relax rule (4) does 26-equal show some promise. Two diatonic scales 2 or 13 degrees apart can be interlaced to satisfy (1a) or (1b). The pattern 1, 5, 9, 12 produces ten and twelve consonant tetrads, respectively, out of fourteen. 22-equal exhibits a similar structure when interlaced pentatonic scales are used, the chord pattern being 1, 4, 7, 9. The resulting "decatonic" scales include modes that do satisfy all of our conditions for a tonal scale:

The decatonic scales

Let's step these scales through our diatonicity rules.

(1) 8s+2L=U. Tuning: 22-equal (L=3, s=2).

- (a) is satisfied by the five modes of the symmetrical scale L s s s s L s s s s, since the large steps are a half-octave apart.
- (b) is satisfied by the ten modes of the scale L s s s L s s s s s, where the "pentachord" consists of one large and three small steps, and two small steps span the "leftover" interval.

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[28] Krumhansl erroneously states that the octatonic scale fails property (2). Krumhansl, op. cit., p.42.

[29] Blackwood, Easley. 1991. "Modes and Chord Progressions in Equal Tunings." *Perspectives of New Music* Vol. 29 No. 2 pp. 166-200.

[30] In ratios, the just major scale is 1/1 9/8 5/4 4/3 3/2 5/3 15/8 (2/1), and the just minor scale is 1/1 9/8 6/5 4/3 3/2 8/5 9/5 (2/1). They differ from the usual diatonic scales in that they contain two additional characteristic dissonant intervals, a 40:27 fifth and a 32:27 third. Structurally equivalent scales exist in 15-, 22-, 27-, and 34-tone equal temperament.

(2) The consonant intervals are Q=L+5s, T=L+2s, Q-T=3s, S=2L+6s, S-Q=L+s (or equivalently, Q-S=L+7s), S-T=L+4s (or equivalently, T-S=L+4s). The pattern 1, 4, 7, 9 produces a consonant tetrad 8 times out of 10 in the version (a) scale, and 6 times out of 10 in the version (b) scale.
(3) In the symmetrical scale, there are two chains of Qs with four consonant tetrads each. In the pentachordal scale, there are two chains of Qs with two consonant tetrads each, and the other two consonant tetrads are not in a chain of Qs.
(4) Verification of this property is left as an exercise for the reader.
(5) Removes the symmetrical mode L s s s s L s s s s and the pentachordal modes L s s s L s s s s s, s L s s s L s s s s, s s L s s s s s L s, and s s s L s s s L s s.

<u>Strong Version</u>

(6) The characteristic dissonance of the symmetrical scale is s+s+s+s. This removes all its remaining modes from consideration. The characteristic dissonances of the pentachordal scale are s+s+s+s+s and s+s+s+s. The 5s interval removes modes s s s s s L s s s L and L s s s s s L s s s, which contain a note that is this interval above the 1st and below the 7th degrees, respectively. The 4s interval removes modes s s s s L s s s L s and s L s s s s s L s s, which contain a note that is this interval above the 1st and below the 7th degrees, respectively. The 4s interval would also remove modes s s L s s s L s s s and s s s L s s s s s L, if it didn't form a complete 9-limit pentad with the tonic tetrad in these modes.
(7) Imposes no further restrictions, leaving the s s L s s s L s s s (standard pentachordal major) and s s s L s s s s s L (standard pentachordal minor) modes.

<u>Weak Version</u>

(6) One of the two 4s intervals in the symmetrical scale forms a complete pentad with the tonic tetrad in s s L s s s s L s s (static symmetrical major) and s s s L s s s s L s (static symmetrical minor). One of the 4s intervals is disjoint from the tonic tetrad in s L s s s s L s s s (dynamic symmetrical major) and s s s s L s s s s L (dynamic symmetrical minor). The characteristic interval of the pentachordal scale is 5s, so the standard pentachordal major and minor modes defined above (both dynamic <u>and</u> static), as well as alternate pentachordal major (s L s s s s s L s s) and minor (s s s s L s s s L s) modes (both dynamic), are tonal.

So we are left with the following possible decatonic modes (expressed as degrees of the 22-tone equal-tempered scale):

Standard Pentachordal Major:	0 2 4 7 9 11 13 16 18 20 (22)
Static Symmetrical Major:	0 2 4 7 9 11 13 15 18 20 (22)
Alternate Pentachordal Major:	0 2 5 7 9 11 13 15 18 20 (22)
Dynamic Symmetrical Major:	0 2 5 7 9 11 13 16 18 20 (22)
Standard Pentachordal Minor:	0 2 4 6 9 11 13 15 17 19 (22)
Static Symmetrical Minor:	0 2 4 6 9 11 13 15 17 20 (22)
Alternate Pentachordal Minor:	0 2 4 6 8 11 13 15 17 20 (22)
Dynamic Symmetrical Minor:	0 2 4 6 8 11 13 15 17 19 (22)

named "Major" or "Minor" after the quality of their tonic tetrads. The five types of (1, 4, 7, 9) tetrads found in these scales can be named as follows:

Major: 0 7 13 18
Minor: 0 6 13 17
Augmented: 0 7 14 18
Major-minor: 0 7 13 17
Minor-major: 0 6 13 18

These chords are encountered in the tonal modes at the following positions:

Position	Quality in std. pent. major mode	Quality in std. pent. minor mode	Quality in static symm. major mode	Quality in static symm. minor mode	Quality in alt. pent. major mode	Quality in alt. pent. minor mode	Quality in dyn. symm. major mode	Quality in dyn. symm. minor mode
I (tonic)	Maj	Min	Maj	Min	Maj	Min	Maj	Min
II (antidominant)	Aug	Ma-mi	Maj	Maj	Maj	Mi-ma	Aug	Min
III (antisubmed.)	Aug	Maj	Aug	Maj	Min	Maj	Min	Maj
IV (mediant)	Min	Maj	Min	Aug	Min	Aug	Min	Maj
V (subdominant)	Ma-mi	Min	Min	Min	Mi-ma	Aug	Maj	Aug
VI (antitonic)	Maj	Min	Maj	Min	Maj	Min	Maj	Min
VII (dominant)	Maj	Mi-ma	Maj	Maj	Aug	Ma-mi	Aug	Min
VIII (submed.)	Min	Maj	Aug	Maj	Aug	Maj	Min	Maj
IX (antimediant)	Min	Aug	Min	Aug	Min	Maj	Min	Maj
X (antisubdom.)	Mi-ma	Aug	Min	Min	Ma-mi	Min	Maj	Aug

The symmetrical scale is useful as a basis for a keyboard mapping of 22-equal. To wit: leaving every "E" on the standard keyboard out of the mapping gives two keyboard octaves to one acoustical octave. The black keys then form a symmetrical scale, which can be thought of as the "natural" scale. Four different pentachordal scales can be played with only one "accidental" -- that is, replacing one out of the ten black keys with a neighboring white key. We can use the numbers 1 through 9 and 0 for the "natural" notes, and the symbols Δ, ∇, and ◊ to indicate chromatically raised, lowered, and natural notes. In the following table, we introduce names for intervals in, and notation for, the decatonic scale, in analogy with ordinary terminology and notation. A subscript of "10" is a reminder that the intervals are measured with respect to the decatonic scale, as opposed to the customary diatonic measurements. The "key signatures" for the tonal decatonic modes are given in the Appendix.

Table 1

Interval in cents	Decatonic interval	Name / Keymap Starting at C#	Name / Keymap Starting at G#	Approx. ratio	Cents for ratio
0	**perfect 1$^{st}_{10}$**	1 / C#	4 / G#	**1:1**	**0**
54.5	aug. 1$^{st}_{10}$ dim. 2$^{nd}_{10}$	1Δ / D 2∇ / D	4Δ / A 5∇ / A		
109.1	minor 2$^{nd}_{10}$	2 / D#	5 / A#	18:17 17:16 16:15 15:14	99.0 105.0 111.7 119.4
163.6	maj. 2$^{nd}_{10}$ dim. 3$^{rd}_{10}$	2Δ / F 3∇ / F	5Δ / B	12:11 11:10 10:9	150.6 165.0 182.4
218.2	**minor 3$^{rd}_{10}$**	3 / F#	6∇ / c	9:8 17:15 **8:7**	203.9 216.7 **231.2**
272.7	**major 3$^{rd}_{10}$** dim. 4$^{th}_{10}$	3Δ / G 4∇ / G	6 / c#	**7:6** 20:17	**266.9** 281.4
327.3	aug. 3$^{rd}_{10}$ **minor 4$^{th}_{10}$**	 4 / G#	6Δ / d 7∇ / d	**6:5** 17:14 11:9	**315.6** 336.1 347.4
381.8	**major 4$^{th}_{10}$**	4Δ / A 5∇ / A	7 / d#	**5:4**	**386.3**
436.4	aug. 4$^{th}_{10}$ dim. 5$^{th}_{10}$	 5 / A#	7Δ / f 8∇ / f	14:11 9:7 22:17	417.5 435.1 446.4
490.9	**perfect 5$^{th}_{10}$**	5Δ / B	8 / f#	**4:3**	**498.0**
545.5	aug. 5$^{th}_{10}$ dim. 6$^{th}_{10}$	 6∇ / c	8Δ / g 9∇ / g	15:11 11:8	537.0 551.3
600.0	**perfect 6$^{th}_{10}$**	6 / c#	9 / g#	**7:5** 24:17 17:12 **10:7**	**582.5** 597.0 603.0 **617.5**
654.5	aug. 6$^{th}_{10}$ dim. 7$^{th}_{10}$	6Δ / d 7∇ / d	9Δ / a 0∇ / a	16:11 22:15	648.7 663.0
709.1	**perfect 7$^{th}_{10}$**	7 / d#	0 / a#	**3:2**	**702.0**
763.6	aug. 7$^{th}_{10}$ dim. 8$^{th}_{10}$	7Δ / f 8∇ / f	0Δ / b	17:11 14:9 11:7	753.6 764.9 782.5
818.2	**minor 8$^{th}_{10}$**	8 / f#	1∇ / c’	**8:5**	**813.7**
872.7	**major 8$^{th}_{10}$** dim. 9$^{th}_{10}$	8Δ / g 9∇ / g	1 / c#’	18:11 28:17 **5:3**	852.6 863.9 **884.4**
927.3	aug. 8$^{th}_{10}$ **minor 9$^{th}_{10}$**	 9 / g#	1Δ / d’ 2∇ / d’	17:10 **12:7**	918.6 **933.1**
981.8	**major 9$^{th}_{10}$** dim. 10$^{th}_{10}$	9Δ / a 0∇ / a	2 / d#’	**7:4** 30:17 16:9	**968.8** 983.3 996.1
1036.4	aug. 9$^{th}_{10}$ minor 10$^{th}_{10}$	 0 / a#	2Δ / f’ 3∇ / f’	9:5 20:11 11:6	1017.6 1035.0 1049.4
1090.9	major 10$^{th}_{10}$	0Δ / b	3 / f#’	28:15 15:8 32:17 17:9	1080.6 1088.3 1095.0 1101.0
1145.5	aug. 10$^{th}_{10}$ dim. 11$^{th}_{10}$	 1∇ / c’	3Δ / g 4∇ / g		

In analogy with the linked 3-limit, or “suspended,” triad of the diatonic scale, the decatonic scales contain linked 3-limit and linked 5-limit, as well as incomplete 9-limit, chords which have four notes and resolve to the 7-limit tetrads. These are:

Linked 3-limit:[31]	0 9 13 18 resolves to major
	0 4 13 17 resolves to minor
Linked 5-limit:[32]	0 7 13 20 resolves to major
	0 6 13 19 resolves to minor
Incomplete 9-limit:[33]	0 5 13 18 resolves to major
	0 8 13 17 resolves to minor

In analogy with the linked 5-limit, or “major and minor seventh,” chords, we may find pairs of tetrads in the decatonic scales that have two notes in common. Unfortunately, each chord has two, rather than one, unshared notes, resulting in four, rather than one, potentially dissonant intervals. The relevant six-tone chords possible in decatonic scales are:

Major+major:	0 2 7 11 13 18
Minor+minor:	0 2 6 11 13 17
Major+minor:	0 2 7 13 18 20
	0 2 6 13 17 19
	0 2 7 9 13 18
	0 5 7 11 13 18

Perhaps a better analogue to the diatonic minor and major seventh chords is found through the “stacking” paradigm. Stacking a 3^{rd}_{10} on a tetrad leads to a duplication of the root, and stacking another 3^{rd}_{10} (as it occurs in the scale) results in the following five-tone chords:

Major+maj. 3^{rd}_{10}:	0 7 13 18 (22) 27
Major+min. 3^{rd}_{10}:	0 7 13 18 (22) 26
Minor+min. 3^{rd}_{10}:	0 6 13 17 (22) 26
Aug.+maj. 3^{rd}_{10}:	0 7 14 18 (22) 27
Ma-mi+min. 3^{rd}_{10}:	0 7 13 17 (22) 26
Mi-ma+min. 3^{rd}_{10}:	0 6 13 18 (22) 26

These may be considered “safe” additions to the tetrads because they add neither ambiguity nor much dissonance, but merely provide “color,” to the tetrads arising in a scale.

[31]Left unresolved, these chords form the decatonic equivalent of “quartal” or “quintal” harmony. Linking three or four Qs in 22-equal leads to good approximations of 7:6 and 9:7, as opposed to the 6:5 and 5:4 that result in a meantone tuning.

[32]These chords, resembling “major and minor seventh” chords, have scalar template 1, 4, 7, 10 and may prove fruitful as an alternate harmonization of the decatonic scale.

[33]Though incomplete in that they do not contain all of the 9-limit intervals, these chords are *saturated*; i.e., no note can be added without introducing an interval beyond the 9-limit. The 0 6 13 19 chord is a saturated 9-limit tetrad as well.

Micro-chromaticism

The author hears the standard pentachordal modes as most stable and most likely to define key centers and modulatory practice. The tonic chords of the alternate pentachordal modes may simply serve as points of intermediate harmonic stability within the standard pentachordal mode. The symmetrical modes have a weird, bitonal quality due to their symmetry at the half-octave. However, it is worth noting that one can consider all the modes above as arising from a major mode with mutable 3rd and 8th degrees and a minor mode with mutable 5th and 10th degrees. When only one of the two mutable degrees is allowed to vary, consonant tetrads can be formed at nine positions in the scale; when either is allowed to vary, consonant tetrads can be formed at all ten positions in the scale. This consideration may indicate two chromatic alterations that will be common when the key center is not changing.

Let us assume the standard pentachordal modes do clearly define the key centers. Moving between one key and another (of the same quality) a Q away often allows short chromatic passages in diatonic music, and the same holds true with the decatonic scale. The resulting pattern of steps is 2 2 3 2 2 2 2 1 1 1 2 2 in 22-equal, allowing for a tonal harmonization of four consecutive notes of the tuning. A greater degree of chromaticism is often achieved by combining keys of both qualities on the same root (parallel keys). The decatonic result is 2 2 2 1 2 2 2 2 1 1 1 1 1 2, thus allowing a tonal harmonization of a micro-chromatic passage six notes long. In 12-equal, total chromaticism is achieved by combining major and minor diatonic modes on the root, the Q above, and the Q below. The decatonic analogue leads to only 18 of the 22 chromatic notes. But adding the immediate micro-chromatic neighbors of the root and the 3:2 above fills in the gaps. Since the root and the 3:2 above are structurally the most important notes, their immediate neighbors can be given an important ornamental role. The result is a tonal framework for total 22-tone micro-chromaticism.

It is well worth noting that just as the 5-limit (diatonic) scale is the complement of the 3-limit (pentatonic) scale in 12-equal, so the complement of the 7-limit (decatonic) scale in 22-equal may serve as a basis for a 9-limit (dodecatonic) system. Figure 1 shows the accuracy of 22-equal in the 9-limit as slightly greater than that of 12-equal in the 5-limit. The interested reader is encouraged to discover how closely the "hexachordal" and symmetrical dodecatonic scales satisfy the requirements of a 9-limit tonal system. Note that the definition of "root" may need modification, since complete pentads will contain two Qs (a 3:1 and a 9:3). The union, starting from the same root, of the four major (or minor) decatonic modes described above is a symmetrical dodecatonic scale.

The hexachordal dodecatonic scale contains within it seven consecutive notes in a cycle of Qs. This forms a diatonic scale where the three "minor" triads are tuned 9:7:6 and the three "major" triads are tuned 1/6:1/7:1/9 – a reversal of the usual harmonic/subharmonic distinctions. For example, the "minor" mode of this scale, which could be termed "sub-minor" due to its small minor thirds, would be 0 4 5 9 13 14 18. This scale can be mapped to the white notes of a keyboard, with the remainder of the hexachordal dodecatonic scale mapped to the black keys. Then the white-note scale satisfies all the properties of the ordinary diatonic scale except that the triads are not complete sonorities (although they can all be partially completed since the 8 or 1/8 is always a scale tone, the required pattern of scale steps is not the same for the minor and major cases), and the dorian and mixolydian modes can no longer support a static tonality.

Table 1 shows that all ratios involving 9, 11, 15, and 17 – in addition to 1, 3, 5, and 7 – can be expressed consistently (and with a maximum error of 20.1 cents) in 22-equal, promising a great wealth of harmonic possibility when "micro-chromaticism" is explored.[34] The harmonies involving 11 should

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[34]This also allows the minor tetrad to be interpreted as 10:12:15:17, for those skeptical of subharmonic interpretations.

prove particularly novel, since they cannot be expressed in 12-equal (in fact, 22 is the simplest equal division of the octave capable of expressing all ratios through the 11-limit consistently – see footnote 8 for a definition of consistency). One scale with which one may begin adding 11-limit flavors has step sizes 4 3 3 3 3 3 3. It contains four 5-limit consonant triads, three octave species with pairs of identical tetrachords, and in this mode, the tonic 4:5:6 triad can be supplemented with a 9 and an 11 – almost a complete 11-limit hexad.

Finally, it is worth noting that 11-tone equal temperament, contained within 22-equal, is a ridiculously dissonant tuning,[35] containing hardly any tonal (root-defining) sounds, and none whatsoever within the 5-limit. The serial composer who is willing to subtract one from the number of notes in the row could be freed from the constant effort to avoid those intervals in 12-equal which define a key center.

Altered scales

Tonal music utilizes not only diatonic scales in their pure form but also altered modes in which one note is displaced from its diatonic position. A general derivation of such scales is to replace one step size with another of a sort found in the unaltered scale. Either the lower or upper note comprising the step may be moved in this operation. The altered modes thus produced must observe rules (2) through (7), with the following condition: If a certain number of scale steps produces an interval that the same number of scale steps never produces in the unaltered scale, that interval will be largely avoided and so will not "count" for rules (6) and (7); i.e. the characteristic dissonances and allowed step sizes remain what they were in the unaltered case. Let's work out the implications for the scales we have already discussed.

<u>Altered diatonic scales</u>

The unaltered scale can be represented as L L s L L L s. Replacing a small step with a large one, there is only one new scale that can be produced:

i) L s L L L L s

The rest of the scales result from replacing a large step with a small one:

ii) s (2L-s) s L L L s
iii) (2L-s) s s L L L s
iv) L L s s (2L-s) L s
v) L L s (2L-s) s L s
vi) L L s L s (2L-s) s
vii) L L s L (2L-s) s s

(2) removes scales (iv) and (vii), where the pattern 1, 3, 5 produces only three consonant triads.
(3) does not eliminate any more scales.
(4) eliminates scales (ii) and (iii), each of which has a note not belonging to any of the consonant triads.
(5) removes scale (i) modes L L L L s L s, L s L s L L L, and s L s L L L L; scale (v) modes L L s (2L-s) s L s, s L s L L s (2L-s), and s L L s (2L-s) s L; and scale (vi) modes L s L s (2L-s) s L, (2L-s) s L L s L s, and s L L s L s (2L-s).

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[35] Of all equal tunings, Blackwood says, "The most effective one for random dissonance is eleven notes." Keislar, Douglas. 1991. "Six American Composers On Nonstandard Tunings." *Perspectives of New Music* Vol. 29 No.1 p. 177.

Strong Version

(6) removes scale (i) modes s L L L L s L and L L L s L s L; the scale (v) modes s (2L-s) s L s L L, L s (2L-s) s L s L, and (2L-s) s L s L L s; and the scale (vi) modes s L s (2L-s) s L L, s (2L-s) s L L s L, and L s L s (2L-s) s L; these modes all contain an augmented fourth below the 5th or above the 1st scale degree. The scale (i) modes L s L L L L s and L L s L s L L both contain a note an augmented fourth from the 3rd scale degree, but it forms a rough 7-limit tetrad with the respective tonic triads.

(7) imposes no further restrictions, leaving the usual harmonic minor (L s L L s (2L-s) s) and harmonic major (L L s L s (2L-s) s) modes, and possibly the melodic minor ascending (L s L L L L s) as well a less common mode known in Carnatic music as Mela Carukesi (L L s L s L L) and sometimes referred to simply as "the Hindu scale."[36] The first two are dynamic, and the last two are both dynamic and static.

Weak Version

(6) removes scale (i) modes s L L L L s L and L L L s L s L, scale (v) modes s (2L-s) s L s L L and L s (2L-s) s L s L, and scale (vi) modes s (2L-s) s L L s L and L s L s (2L-s) s L, all of whose tonic triads share a note with each of the two augmented fourths in their respective scales. This leaves, in addition to the strongly tonal modes above, the mode known to the Hindus as Mela Kosalam ((2L-s) s L s L L s), and a mode with no recognized name, but which jazz musicians might describe as phrygian flat 4 (s L s (2L-s) s L L),[37] both of which are dynamic. If rough 7-limit tetrads are allowed, we also have phrygian sharp 6 or Mela Natakapriya (s L L L L s L) and lydian dominant or Mela Vacaspati (L L L s L s L), both of which are static.

Altered pentatonic scales

The unaltered scale can be represented as s s L s L. Replacing a large step with a small one, there is only one new scale that can be produced:

i) s s s L L

The rest of the scales result from replacing a small step with a large one:

ii) L (2s-L) L s L

iii) (2s-L) L L s L

(2) removes all three scales, since the pattern 1, 4 produces only two consonant dyads in any of them.

Altered decatonic scales

The reader may verify that there are none.

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[36]It should be noted that most of the considerations leading to these scales are foreign to Indian classical music, which uses these scales purely melodically over a tonic drone. These four modes are known, in Schenker's system, as Mischungen 4, 2, 1, and 6, respectively. Schenker, op. cit.

[37]A comprehensive list of over 300 mode names did not include this mode until the author suggested its addition. The Indian names are taken from this list. Op de Coul, Manuel, ftp://ella.mills.edu/pub/ccm/tuning/papers/modename.txt.

History of 22-tone tunings

Indian music is described in terms of a 22-*sruti* division of the octave. Whether this represents an actual set of pitches, or just a means of specifying that some scale steps were to be larger than others, is controversial. Some readings of early accounts suggest a tuning approaching 22-tone equal temperament.[38] Modern accounts describe the three step sizes of the basic seven-tone Indian scales by taking 4-*sruti* intervals as 9:8, 3-*sruti* intervals as 10:9, and 2-*sruti* intervals as 16:15. This determines 3:2 intervals as 13 *srutsi*, 4:3 as 9 *srutis*, and 5:4 as 7 *srutis*. Thus the harmonic structure of these scales is well represented in 22-equal. But the melodic structure can be somewhat different.

The two basic scales specified in ancient Indian theory are *sa-grama* and *ma-grama*.[39] Their structure was described according to the following table (notes named with a + sign were chromatic tones recognized by the ancients; the derivation of the ratios is discussed below):

Table 2

sruti	position in *sa-grama*		position in *ma-grama*	
1				
2	Ni+	3/√2 (tempered 16/15)	Ga+	√2 (tempered 45/32)
3				
4	Sa	9/8	Ma	3/2
5				
6		(6/5)		[8/5]
7	Ri	5/4	Pa	5/3
8				
9	Ga	4/3		[16/9]
10				
11	Ga+	45/32	Dha	15/8
12				
13	Ma	3/2	Ni	1/1
14				
15		(8/5)	Ni+	16/15
16				
17	Pa	27/16	Sa	9/8
18				
19		(9/5)		[6/5]
20	Dha	15/8	Ri	5/4
21				
22	Ni	2/1	Ga	4/3

If the *srutis* are indeed 22-equal, the total set of *sruti* positions for which names are given in table 2 form a pentachordal decatonic scale! It is therefore conceivable that some of the melodic patterns specified in Indian music theory[40] may be useful for decatonic composition, especially if they happen to outline parts of tetradic structures.

One plausible just interpretation of this diagram is that two parallel strings on an instrument were tuned a 4:3 apart, the *sruti* numbers referred to fret positions, and the note names were meant to refer to practically the same absolute pitches on both strings: a set of pitches approximating a diatonic scale.

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[38] Chakraborty, Satyaban. 1990. *Svaras and Srutis in Indian Music*. Kalyani Prakashani, West Bengal.

[39] A third scale, *ga-grama*, was reputedly of Himalayan origin and fell out of use at an early stage.

[40] Rowell, Lewis. 1992. *Music and Musical Thought in Early India*. University of Chicago Press.

Another possible interpretation is that the diagram refers to two methods of playing a single string, and it is the *sruti* numbers that referred to a fixed set of pitches, while the note names conveyed relative positions within a mode. The note names were shifted by one scale degree in later Indian theory, so that Ni is currently known as Sa, etc., but the names of the *gramas* have not changed. The earlier terminology is used in table 2 and henceforth. The unaltered *sa-grama* tones fixed seven of the positions on the instrument. Ratios are added to the diagram in accordance with the modern, 5-limit specification of the step sizes of the unaltered sa-*grama*, with Ni taken as 1/1 since it corresponds to an open string and to the modern tonic Sa.

Of those seven positions, five could be used in playing notes Ni, Sa, Ri, Ga, and Ma of *ma-grama*. The *sruti* corresponding to note Ri of *sa-grama* would, in *ma-grama*, produce a Pa 21.5 cents shy relative to note Pa of *sa-grama*. Rather than correcting this discrepancy by fixing another *sruti* position at such a short distance from an established one, Pa was recognized in two forms, and *sa-grama* and *ma-grama* became the names of the diatonic scale with the two different versions of Pa. Finally, in order to play note Dha of *ma-grama*, an entirely new position had to be added, 92.2 cents above the position corresponding to corresponding to note Ga of *sa-grama*. (This tuning is necessary in the just specification since Pa-Dha is a step of 4 *sruti*, so must be tuned 9:8, in *ma-grama*.)

This new position corresponded to a chromatic alteration, Ga+, on the first string, that may have been added as a result of this process or may have been previously recognized. The former implies a tuning of 45/32; in the latter case, tuning it as a 5-limit interval from an existing scale tone can again only produce 45/32. Note that the just specification of the step sizes alone is not sufficient to determine the tuning of such chromatic alterations: the 4-*sruti* interval of 9:8, minus a 2-*sruti* interval of 16:15, leaves another 2-*sruti* interval of 135:128, which could conceivably occur either above or below the chromatic note. The other chromatic alteration recognized by early Indian theory was Ni+. Modern theory takes the tuning of this note as 16/15. This seems the most likely tuning of the note in *ma-grama*, as it is a 5:4 below the open string of that *grama*. Fixing the position of *sruti* 2 in accordance with this tuning in *sa-grama* would produce a *ma-grama* Ga+ of 64/45, while consistently using 45/32 for Ga+ would force *sruti* 2 to give a *sa-grama* Ni+ of 135/128. So that a single *sruti* position could serve both functions, there is evidence that a compromise between the two was adopted.[41] In more recent theory, other altered notes have been added to *sa-grama*, their simple ratios shown in parentheses in table 2. The 12 *sruti* positions in the *sa-grama* for which ratios have been given are known as the Modern Indian Gamut.[42] If the *sruti* are taken as equally tempered, the Modern Indian Gamut is a hexachordal dodecatonic scale.

But how did the particular quantification in terms of *sruti* come about? If we take the 5-limit just interpretations seriously, it is difficult to see the *sruti* as units of a specific size, since the difference between the 4- and 3-*sruti* intervals is virtually identical to the difference between the two 2-*sruti* intervals.[43] One clear feature of Indian music is the "drone" consisting of 1/1 and 3/2. The conventional modern enumeration of the *sruti* tunes five consecutive 3:2s above the 3/2 and five below the 1/1, forming a Pythagorean chain of 12 notes. The other ten notes form five just major and five just minor triads with the six central notes of the Pythagorean chain.[44] Depicting 3:2 relationships as proceeding

[41]The fret corresponding to *sruti* 2 was typically placed exactly halfway between the nut and the fret for *sruti* 4 (this would produce a harmonic mean frequency, 3 cents different from the geometric mean we have assumed); and all 9- or 13-*sruti* intervals were considered consonant. Chakraborty, op. cit.

[42]Deva, B. Chaitanya. 1980. *The Music of India: A Scientific Study.* Munshiram Manoharlal, New Delhi.

[43]The difference between the differences is a schisma, or less than 2 cents.

[44]See for example Ramanathan, S. Feb. 1973. "The Sruti Veena," in *Musical Scales: Report of Symposium.* Sangeet Natak Akademi, New Delhi.

rightwards, 5:4 relationships upwards and to the right, 5:3 relationships upwards and to the left, the ancient gamut in bold, and the modern additions in bold Italics, the *srutis* can be depicted as follows:

Table 3

sa-grama (1/1=0, 3/2=13):

				3		16		7		20		11				
1	14	5	18		9		0		13		4		17	8	21	12
				2		15		*6*		*19*		10				

ma-grama (1/1=13, 3/2=4):

				16		7		20		11		2				
14	5	18	9		0		13		4		17		8	21	12	3
				15		*6*		*19*		10		1				

The *sruti* numbers simply order the tones according to pitch, with the drone tones numbered as in Table 2. Note that the tuning of the *sruti* numbered 2 depends on the choice of *grama*, motivating the compromise discussed above. Indian theorists could have originally derived these pitches independently of the sruti idea by simply tuning ten consecutive 3:2s above the 3/2 and ten more below the 1/1, which would result in virtually the same pitches.[45] An additional 3:2 on either end of the chain would result in a note a Pythagorean comma (23.5 cents) from a drone note, which may be why the theorists stopped at 22.

Kraehenbuehl and Schmidt, inspired by Yasser, proposed an evolution proceeding from pentatonic to heptatonic to chromatic (12 tones) to "hyperchromatic" (22 tones) to 41 tones per octave, corresponding with a harmonic limit that increases from 3 to 5 to 7 to 11. They defined all the tones with just ratios and defined the harmonic limit as the highest *prime* number used in these ratios (thus they skipped 9). The evolution proceeds by interpolating one new note into each of the larger steps of the previous system, first using only the older harmonic limit. Then, once the newer system begins to be used in its entirety, the next harmonic limit begins to cause "inflections" in the tuning of the older system to occur. Once the newer harmonic limit takes effect, the process begins again.[46] The "included chromatic system of the hyperchromatic system" resembles the symmetrical dodecatonic scale mentioned earlier. Although Kraehenbuehl and Schmidt do not address the issue of temperament, it is reasonable to suppose that they would allow for equal temperament when it preserves the consonance relationships of the prevailing harmonic limit, so that the present usage of 12-equal may be seen according to their theory. Nor do Kraehenbuehl and Schmidt address the relationships between chord structure and scale structure, perhaps because in just intonation, the number of inflections required to produce consonant chords on most scale degrees grows rapidly as the limit increases.

The author, unaware of these theories, discovered the decatonic scales geometrically in 1991. The diatonic scale can be depicted as in Figure 3. Here the Q and T intervals are represented by solid lines at 60° angles from one another. The remaining 5-limit interval, 6:5, is represented by a dotted line.

[45] The largest error is again a schisma. Using this "Pythagorean" approach would require that sixth 3:2 in each chain (the line breaks in table 3) be reckoned as 12 *sruti*.

[46] Kraehenbuehl, David and Schmidt, Christopher. 1962. "On the Development of Musical Systems." *J. Music Theory* vol. 6 no. 1 pp. 32-65.

The consonant triads of the scale thus form equilateral triangles. The figure has one equilateral triangle for every consonant triad in the scale. “D” appears twice and if all intervals were tuned in just intonation, “D” would require two distinct tunings, differing by 81:80 or 21.5 cents. Only in a meantone tuning, i.e. a tuning for which 4Q=T, do both occurrences of “D” represent the same pitch. To find a similar construct for the 7-limit, we need an additional dimension. Figure 4 depicts a 3-dimensional lattice where the Q, T, and S axes are meant to be at 60° angles from one another. The viewpoint is 1° off perpendicular to the Q-T plane. The solid lines are all parallel to the axes. Dotted lines are added to represent the other consonant intervals: 6:5, 7:5, 7:6. Each obliquely distorted “unit cube” of the solid-line lattice then consists of two regular tetrahedra and one regular octahedron. The tetrahedra are the complete, consonant tetrads of 7-limit harmony, the major tetrad appearing “right-side-up” and the minor tetrad “upside-down.”[47] The vertices of the lattice are labeled with numbers from 0 through 21 representing 22-tone equal temperament. For example, moving one step in the positive direction along the Q axis corresponds to an addition of 13, the T axis an addition of 7, and the S axis an addition of 18, all mod 22. Figure 5 shows a portion of this lattice that contains the six consonant tetrads of the pentachordal decatonic scale; note that two notes appear twice and would thus each need two distinct tunings, differing by 50:49 or 35.0 cents, in just intonation. Figure 6 is a portion of figure 4 showing the symmetrical decatonic scale; its eight consonant tetrads would require that four notes exist in two versions a 50:49 apart, and two additional notes have two versions a 64:63, or 27.3 cents, apart, in just intonation. While figure 3 is the representation of the triads of the diatonic scale with the fewest replicated notes, figures 5 and 6 are only examples and other equally simple arrangements of the decatonic tetrads exist.

It turns out that Ben Johnston has employed a 22-tone 7-limit just tuning in his 4th string quartet (1973). This tuning has been displayed on a just lattice constructed much like figure 4.[48] When its structure is reproduced on the 22-equal lattice, it forms a one-to-one correspondence with 22-equal. Johnston was focusing mainly on harmonies, but was likely guided by a desire to achieve rough equality of the smallest melodic steps. Thus he may have been the first to hint upon the applicability of 22-equal to 7-limit harmony in an actual composition.

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[47]The octahedron is a Wilson hexany, while the oblique cube is an Euler-Fokker genus. Indeed, in any number of dimensions, the “unit hypercube” of the lattice, an Euler-Fokker genus with no repeated factors, can be constructed by “tiling” all Wilson CPSs (where the set of available factors, assumed to be relatively prime, is the set of axes). Fokker, op. cit.; Chalmers, John H. Jr. and Wilson, Ervin M., 1982. "Combination product sets and other harmonic and melodic structures*", Proceedings of the 1981 International Computer Music Conference.* ICMA, 1982, pp. 348-362.

[48]Gilmore, Bob. 1995. “Changing the Metaphor: Ratio Models of Pitch in the Work of Harry Partch, Ben Johnston, and James Tenney.” *Perspectives of New Music* vol. 33 p. 482.

Tuning the decatonic scale

It is well known that the consonances of the diatonic scale are more accurately expressed in varieties of meantone tuning other than 12-equal. Figure 1 shows that the accuracy of 19- and 31-equal in the 5-limit is greater than that of 12-equal, and these are indeed meantone tunings. A meantone tuning is defined completely by specifying the size of the Q. The same is true for the decatonic scale, since it can be constructed entirely from the Q (perfect 7^{th}_{10}) and the half-octave (perfect 6^{th}_{10}). Since any perfect 6^{th}_{10} in the scale is likely to represent 7:5 and 10:7 equally often, the interval achieves its greatest accuracy at exactly 600 cents. Thus it remains only to vary the size of the perfect 7^{th}_{10}. Maximum accuracy, accuracy defined as it is for figure 1, is achieved with a perfect 7^{th}_{10} of $\frac{30.5 + 7 \cdot \log_2(3) - 5 \cdot \log_2(35)}{27}$ octaves, or 708.8143 cents, only 0.3 cents less than the 22-equal value. The smallest equal temperament with an even number of notes (so that there is a half-octave) which better approximates 708.8143 cents is 210-tone equal temperament. Successive improvements come with each addition of 22 to this number up through 386. If accuracy is defined as it is for figure 2, no tuning of the decatonic scale is very good, but a perfect 7^{th}_{10} of $\frac{1186.5 + 231 \cdot \log_2(3) - 125 \cdot \log_2(5) - 245 \cdot \log_2(7)}{971}$ octaves, or 710.0927 cents, is optimal.[49] The smallest even equal temperament which improves upon 22-equal is then 76-equal, and successive improvements come with 98-, 120-, and 218-tone equal temperament.[50] The smallest even equal temperaments not yet mentioned with an interval between these two optimal values, and which are not multiples of 22, come from successive additions of 22 starting with 142. However, in all of these larger tuning systems, the increase in accuracy is negligible, and there are better approximations to some 7-limit intervals than those occurring in the decatonic scale. Therefore, if simplicity is at all an issue, it is clear that 22-equal is the best tuning for the decatonic scale.

One can also construct an unequal 22-tone tuning which allows both decatonic and Indian scales. Five notes of the Modern Indian gamut form a chain of four just Qs from 4/3 to 27/16. Next there is a chain of two just Qs from 5/4 to 45/32, a tempered Q to $3/\sqrt{2}$, another tempered Q to 8/5, and two just Qs to 9/5. Now if we consider the 22 *sruti* as approximating 22-equal, the remaining twelve Qs can form three different pentachordal and two different symmetrical decatonic scales. Since we want the Indian "just" Qs to be as close to just as possible, we make the decatonic Qs as large as possible. With a perfect 7^{th}_{10} any larger than 711 cents, the minor 4^{th}_{10} would possess an error greater than that of the 600-cent perfect 6^{th}_{10}, which must approximate both 7:5 and 10:7 with an error of 17 cents.

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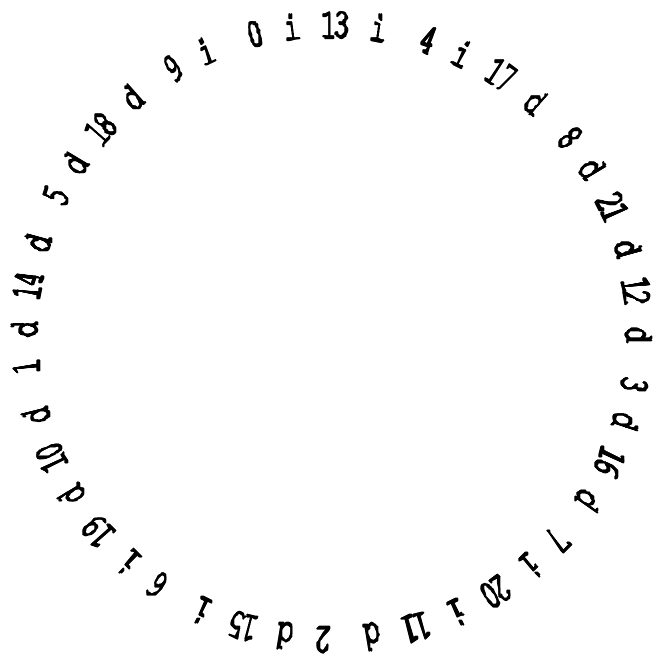
[49]For the diatonic scale, the two optimal values of the perfect fifth are $\frac{2 - 2 \cdot \log_2(3) + 7 \cdot \log_2(5)}{26}$ octaves or 696.1648 cents (fig. 1) and $\frac{66 - 66 \cdot \log_2(3) + 175 \cdot \log_2(5)}{634}$ octaves or 696.0187 cents (fig. 2). These correspond to 7/26-comma meantone and 175/634-comma meantone temperaments, respectively. The former was derived as the optimal diatonic tuning in Woolhouse, W. S. B. 1835. *Essay on Musical Intervals, Harmonics, and the Temperament of the Musical Scale, &c.* J. Souter, London.

[50]76-equal is notable for having 19-equal contained within it. It therefore contains all the tonal systems discussed in this paper (including two distinctly tuned pentatonic systems). A slightly simpler equal tuning with this property is 64-equal, where the accuracy (especially of the minor 4^{th}_{10}) is comparatively poor. These are similar in complexity to 72-equal, which has been used for its extremely close correspondence with 11-limit just intonation. 120-equal is convenient because all intervals are multiples of 10 cents. In 218-equal, the perfect 7^{th}_{10} is 0.0010 cent short of its optimal size with respect to the fig. 2 definition.

Figure 7



Viewing the tuning as a circle of 3:2s, the half-circle that includes the original five notes (top half of figure 7) must have four Indian fifths (represented by “i”) plus seven 711-cent perfect 7^{th}_{10}s (represented by “d”) add up to 600 cents (mod the octave). That gives an Indian Q of 705.75 cents, approximately the 17-equal Q. Now the other half-circle has five 711-cent perfect 7^{th}_{10}s, four Indian 705.75-cent Qs, and what remains represents the two Indian tempered Qs, which must therefore be tuned to 711 cents – equal to the decatonic Q.

A keyboard mapping for this tuning which allows the available decatonic scales to use the simplest key signatures (i. e., few or no white keys) is given in table 4. If desired, the two different versions of the tempered *sruti* 2 may be tuned on the “E” and “F” of the keyboard, as shown in braces in table 4. Columns 4 and 5 of table 4 show exact tunings for the two *gramas* as well as the speculative ratios in parentheses. The tunings are shifted so that they have the same pitch at the center of symmetry. The largest error within the *gramas* is then about 7½ cents.

The key signatures of the available symmetrical decatonic scales in this tuning are {} and {5Δ, 0Δ}; the pentachordal signatures are {6Δ}, {0Δ}, and {5Δ, 9Δ, 0Δ}.

Table 4

keyboard mapping	cents	*sruti* number	just ratios and cents for ***sa-arama*** (3/2 — 705.75)	just ratios and cents for ***ma-arama*** (1/1 — 705.75)	decatonic note name
D	0	0	**1/1 — 3.8**	**4/3 — 3.8**	
D#	50.25	1			2
{E}	{100.5}	{2}		{45/32 — 96}	
E&F	105.75	2	3/√2 — 105.75	√2 — 105.75	
{F}	{111}	{2}	{16/15 — 115.5}		
F#	161.25	3			3
G	211.5	4	**9/8 — 207.7**	**3/2 — 207.7**	
G#	272.25	5			4
A	322.5	6	6/5 — 319.4	8/5 — 319.4	
A#	383.25	7	**5/4 — 390.1**	**5/3 — 390.1**	5
B	428.25	8			5∆
c	494.25	9	**4/3 — 501.8**	16/9 — 501.8	6∇
c#	539.25	10			6
d	594.75	11	45/32 — 594	**15/8 — 594**	
d#	650.25	12			7
e&f	705.75	13	**3/2 — 705.75**	**1/1 — 705.75**	
f#	761.25	14			8
g	816.75	15	8/5 — 817.5	16/15 — 817.5	
g#	872.25	16			9
a	917.25	17	**27/16 — 909.7**	**9/8 — 909.7**	9∆
a#	983.25	18			0
b	1028.25	19	9/5 — 1021.4	6/5 — 1021.4	0∆
c’	1089	20	**15/8 — 1092.1**	**5/4 — 1092.1**	
c#’	1139.25	21			1

Acknowledgments

I wish to thank Ramon Satyendra for serving as advisor to the project for which this paper was originally written. Thanks also to John Chalmers and Manuel Op de Coul for many valuable references and comments.

Question—What would be most helpful in music today to the composer, to the performer, and to the theorizer?

Answer—If we could bring it about, that the diatonic scale be “spurlos versenkt” and that, instead of busying himself with *absorbing* “a” scale (that is, “the” diatonic scale of the past), the music pupil devote his time to *the invention of scales* for his own fun and for entertaining others with music built of the elements of such scales of his invention.

--Max Meyer[51]

[51]Meyer, Max F. 1929. “The Musician’s Arithmetic: Drill Problems for an Introduction to the Scientific Study of Musical Composition.” *University of Missouri Studies* Vol. IV. Colombia, Missouri, p.93.

Appendix: Decatonic Key Signatures

Major and Minor in the first and third columns below refer to the Standard Pentachordal modes. For the Alternate Pentachordal modes, look at the same row in the opposite column.

Decatonic Key Signatures and Rezsutek Keyboard Layout

The key signatures in the first and third columns refer to the standard pentachordal major and minor modes built on the indicated tonics. For the alternate pentachordal major and minor modes, match the key signatures in the first column with the respective tonics in the third column, and vice versa.

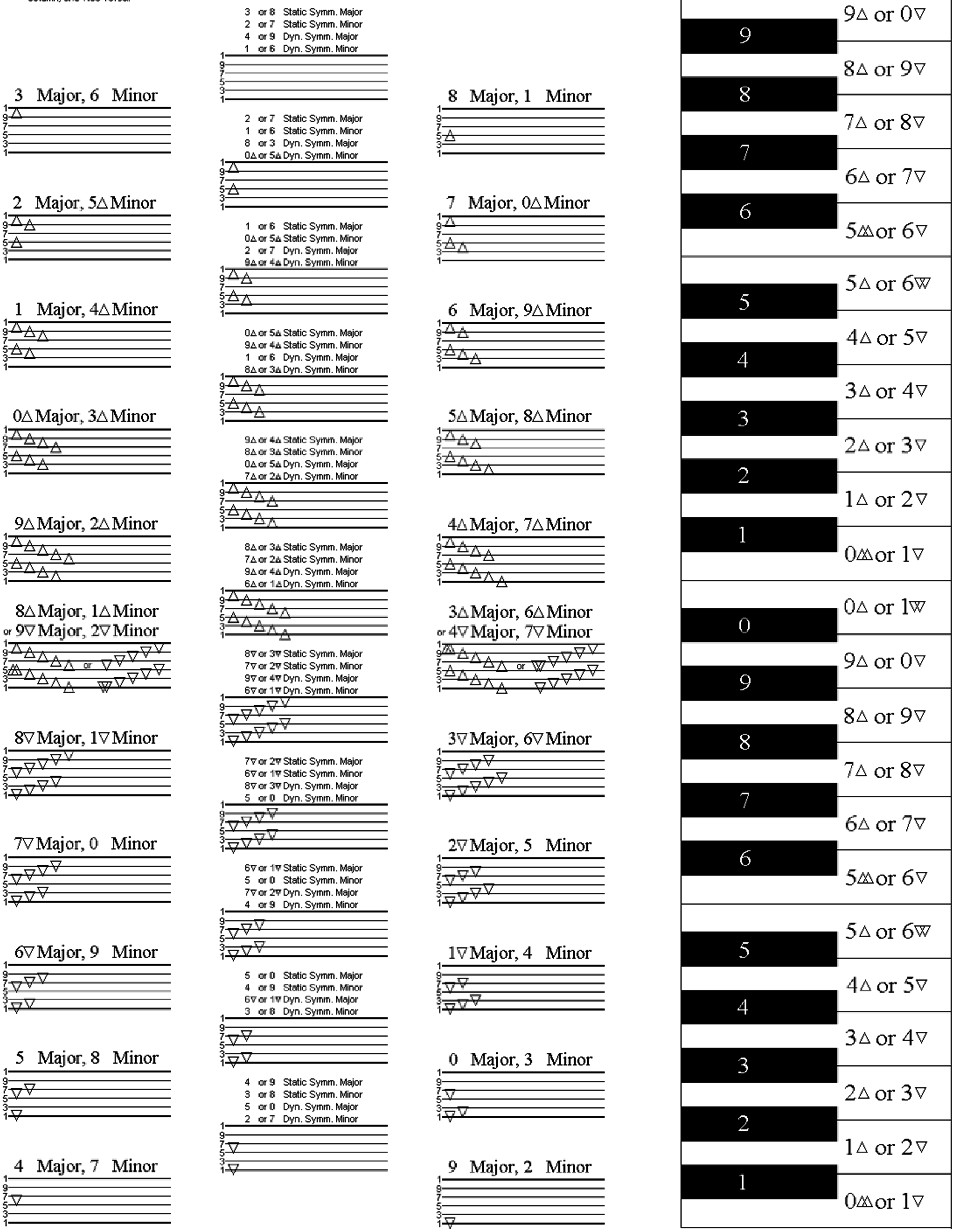
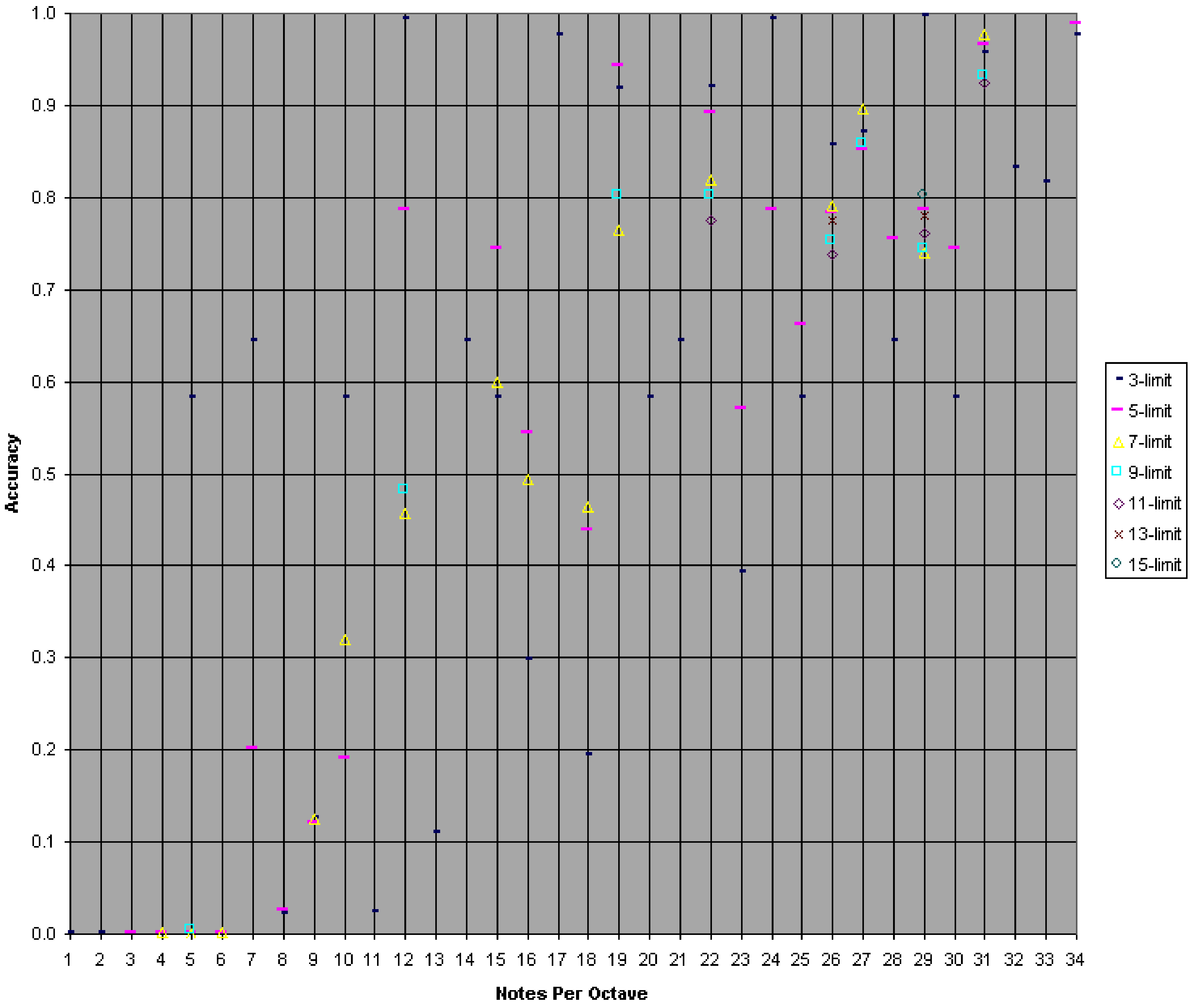


Figure 1



1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
Accuracy
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34
Notes Per Octave
3-limit
5-limit
7-limit
9-limit
11-limit
13-limit
15-limit

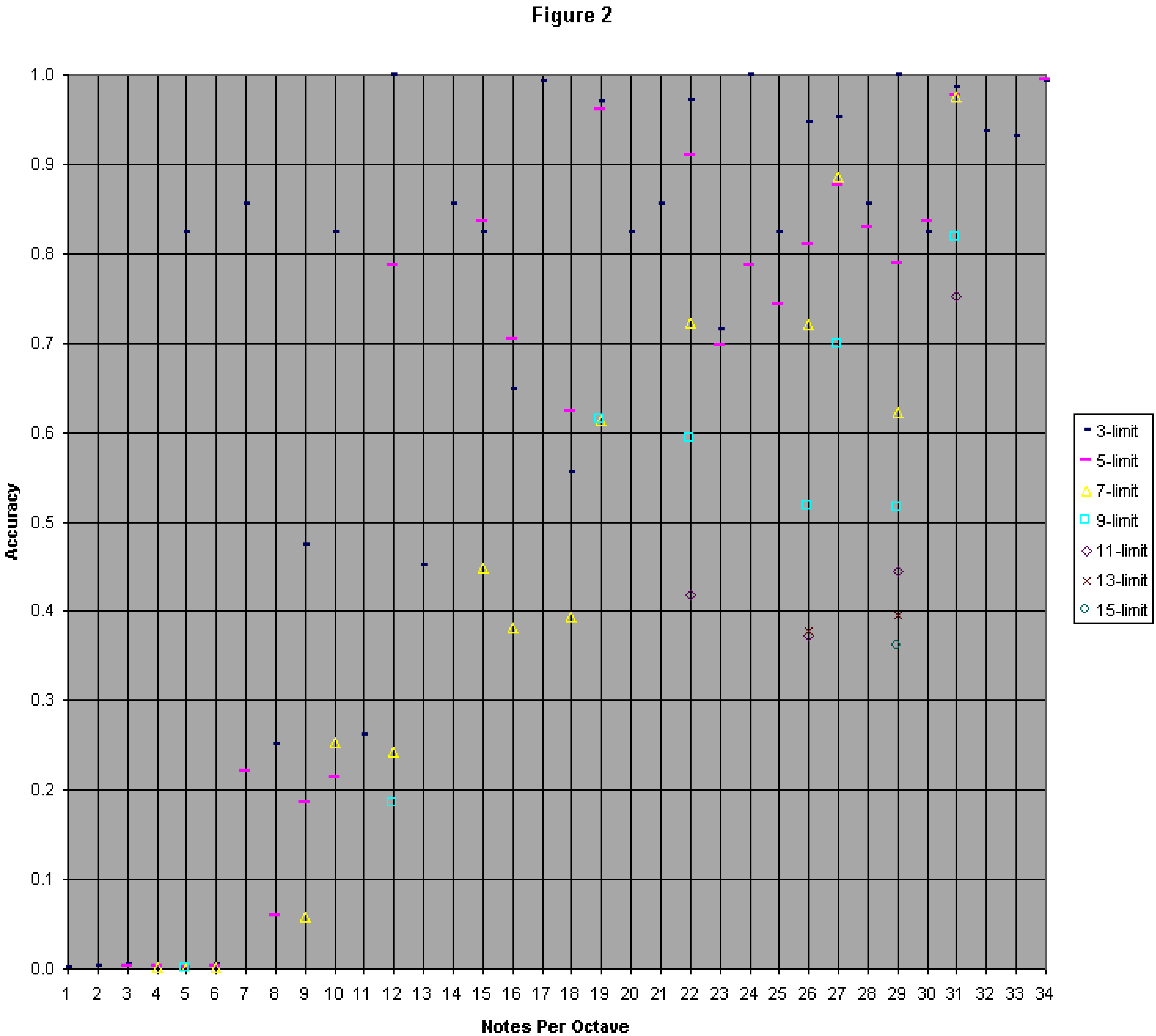


Figure 2
Accuracy
Notes Per Octave
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34
3-limit
5-limit
7-limit
9-limit
11-limit
13-limit
15-limit

Figure 3

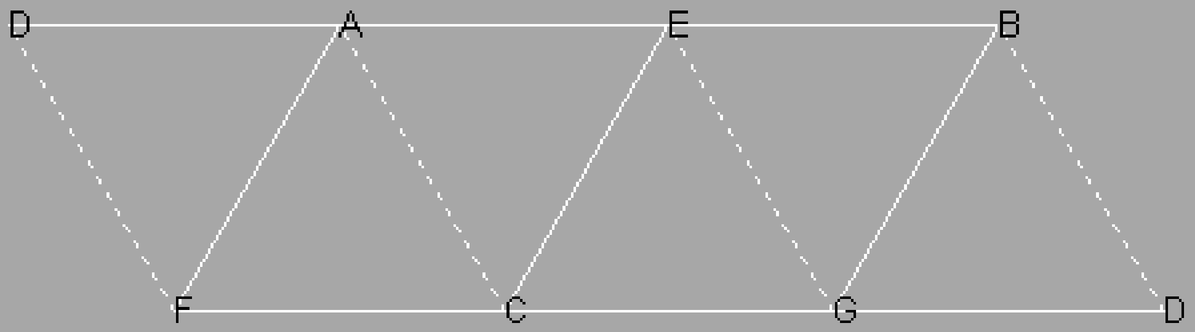
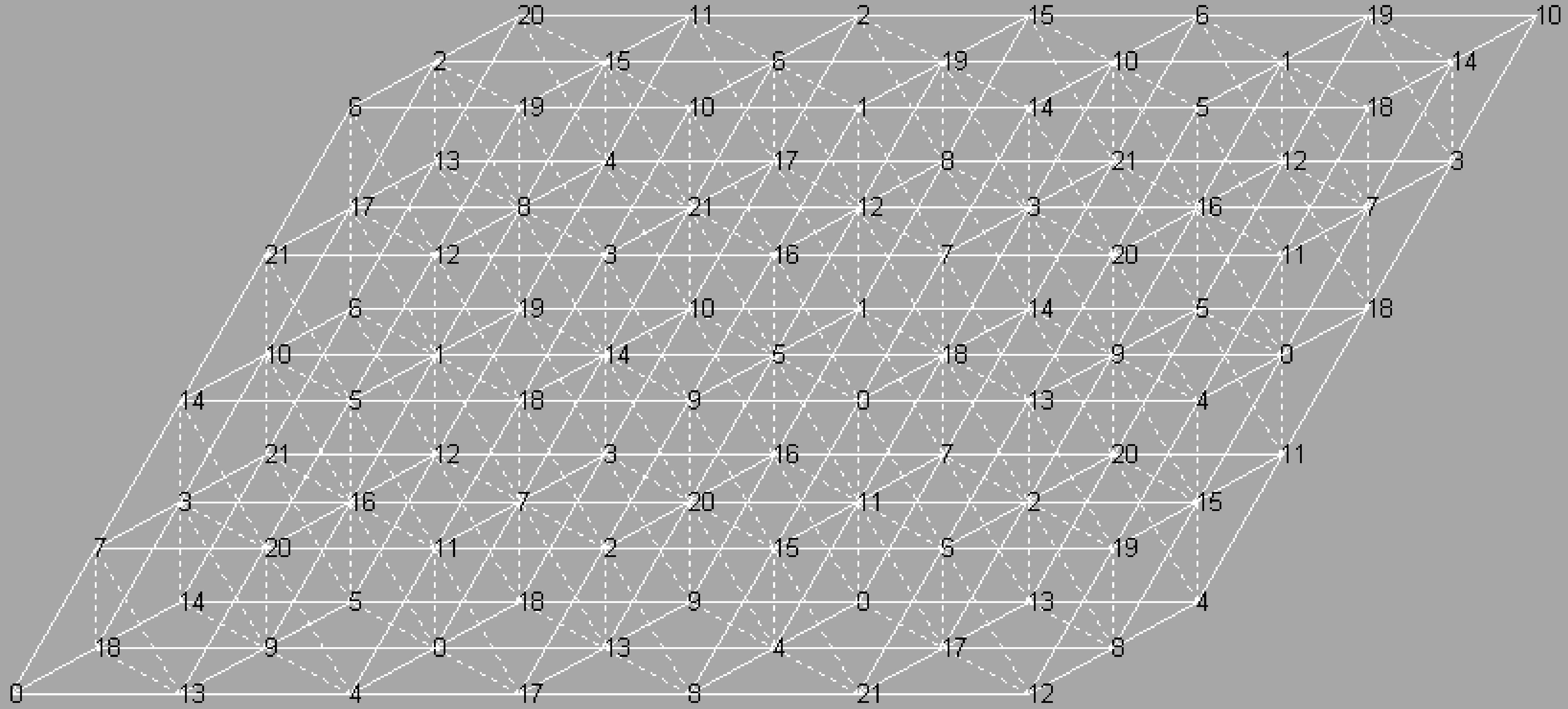


Figure 4



20 11 2 15 6 19 10
2 15 6 19 10 1 14
6 19 10 1 14 5 18
13 4 17 8 21 12 3
17 8 21 12 3 16 7
21 12 3 16 7 20 11
6 19 10 1 14 5 18
10 1 14 5 18 9 0
14 5 18 9 0 13 4
21 12 3 16 7 20 11
3 16 7 20 11 2 15
7 20 11 2 15 6 19
14 5 18 9 0 13 4
18 9 0 13 4 17 8
0 13 4 17 8 21 12

Figure 5

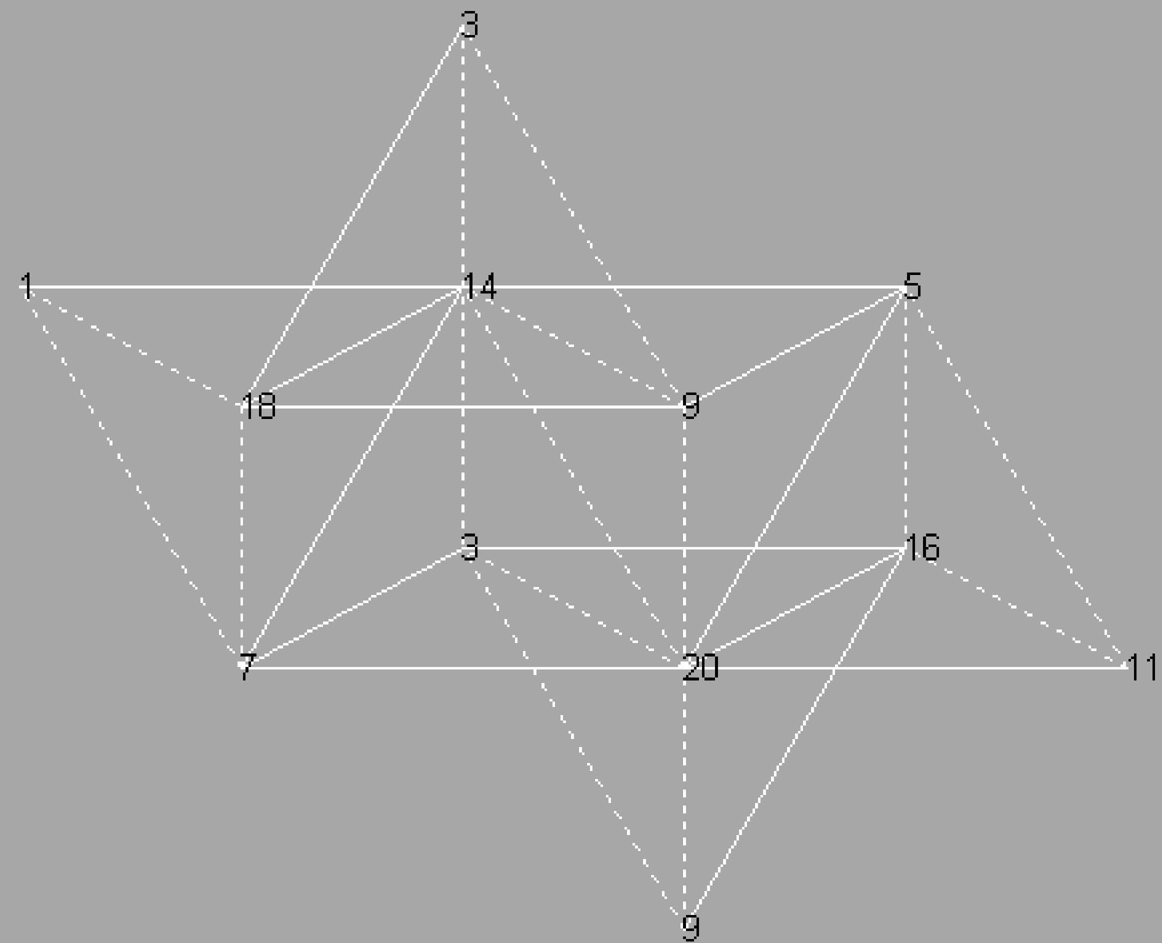


Figure 6