The Forms of Tonality

a preview

This paper briefly illustrates the concepts of tone-lattices, scales, periodicity and notational systems for 5-limit and 7-limit music. These concepts will be explored in much greater detail in a future work.

Consonance and Dissonance (optional)

Using the harmonic entropy model,[1] one obtains general-purpose dyadic dissonance curves such as the example in FIGURE 1. For the simplest ratios, remarkable agreement is found with Tenney's Harmonic Distance function,[2] i.e., the exponential of the dissonance measure is directly proportional to the product of the ratio's numerator and denominator. Between these ratios, we see a continuous curve, similar to those obtained by Helmholtz,[3] Sethares,[4] and others.

However, we may wish to think of intervals in an octave-invariant sense, so that we may construct music theories like that of Partch:[5] the field of pitches to consider is reduced to within one octave. This move reflects many composers' use of techniques relying on the phenomenon of octave-equivalence in pitch perception, and offers great theoretical simplification by obviating the need to evaluate separately many intervals in a given interval-class. Partch classed interval-ratios by an odd number ("Ratios of 3", "Ratios of 5", "of 7", "of 9" . . .) which is defined as the largest odd factor found in either the numerator or the denominator of the ratio. Unlike Partch, we will use these terms to describe ratios when they represent interval classes (dyads -- for which consonance is relevant), but never pitch classes (for which it isn't).

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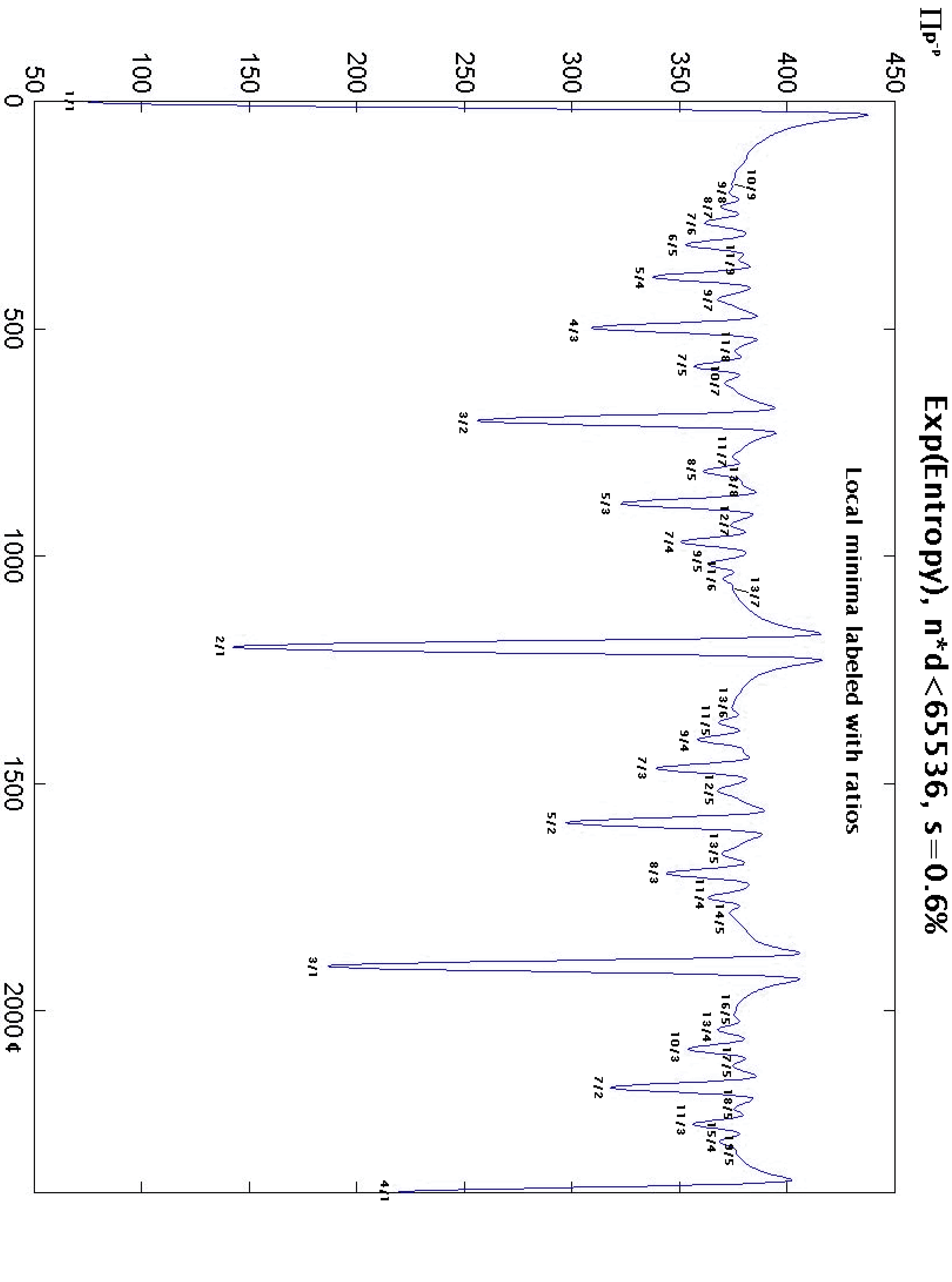
[1] Discussed in detail at http://groups.yahoo.com/group/harmonic_entropy.
[2] Tenney, James. "John Cage and the Theory of Harmony", *Soundings* vol. 13, 1984, pp. 55-83.
[3] Helmholtz, Hermann L.F. von. *On the Sensations of Tone as a Psychological basis for the Theory of Music*. 2nd English edition translated by Alexander John Ellis, based on the 4th German edition of 1877 with extensive notes, foreword and afterword: 1885. Reprint by Dover Publications, 1954.
[4] Sethares, William A. *Tuning, Timbre, Spectrum, Scale*. Springer-Verlag, London, 1998.
[5] Partch, Harry. Genesis of a Music: An Account of a Creative Work, its Roots and its Fulfillments. Da Capo Press, New York, 1974.

FIGURE 1



Exp(Entropy), n*d<65536, s=0.6%
Local minima labeled with ratios
Πp^-p
0
500
1000
1500
2000¢
50
100
150
200
250
300
350
400
450

By "cheating" and building octave-equivalence into the "seeding" of the harmonic entropy model, we obtain octave-equivalent dissonance curves such as the example in FIGURE 2. The ratios found at the local minima of these latter curves follow the same pattern as Partch ascribed to these ratios with his "One-Footed Bride".[6] Ratios of 1 are most consonant, followed by Ratios of 3, then Ratios of 5, and so on, becoming more dissonant in smaller and smaller increments until the ratios become too complex to tune directly by ear (such as Partch's "secondary ratios").

Basic Lattice Structures

The intervallic relationships in a pitch set are most easily seen in a lattice diagram. The diagrams in this paper will use the following conventions for depicting intervals:[7]

Ratios of 3

3:2 or 4:3 –- Red line –- West end represents 2 or 4; east end represents 3.

Ratios of 5

5:4 or 8:5 –- Blue line –- Southwest end represents 4 or 8; northeast end represents 5.

5:3 or 6:5 – Magenta line -– Southeast end represents 3 or 6; northwest end represents 5.

Ratios of 7

7:4 or 8:7 –- Green line –- Northwest end represents 4 or 8; southeast end represents 7.

7:6 or 12:7 – Yellow line – Northeast end represents 6 or 12; southwest end represents 7.

7:5 or 10:7 – Cyan line – North end represents 5 or 10; south end represents 7.

Pitches are represented by white circles, sometimes encircling either a ratio or a notational symbol, signifying pitch class.

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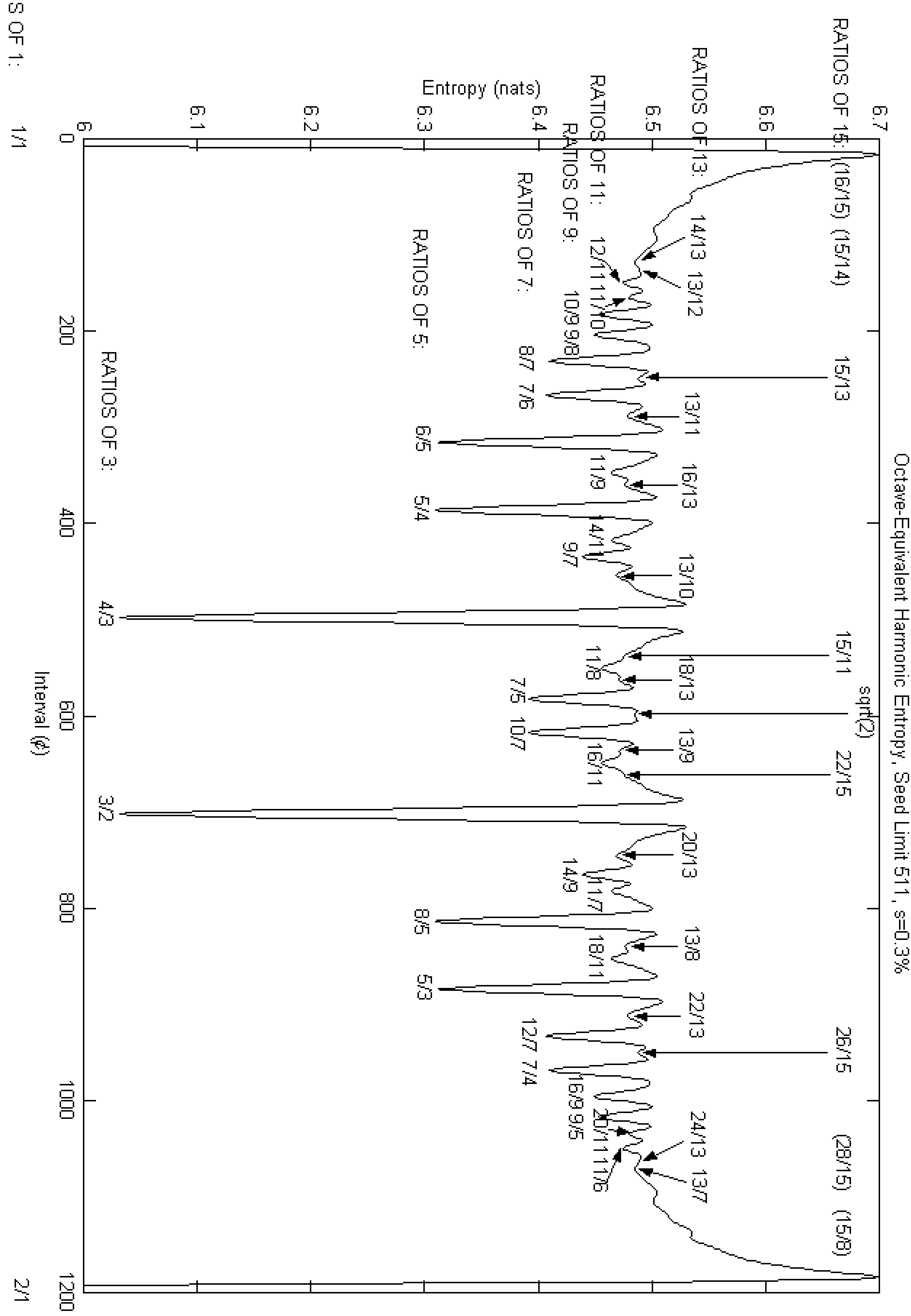
[6] Ibid, p. 155.

[7] The orientations used are taken from Dave Keenan, http://users.bigpond.net.au/d.keenan.

FIGURE 2



Octave-Equivalent Harmonic Entropy, Seed Limit 511, s=0.3%
Entropy (nats)
Interval (¢)
RATIOS OF 1:
1/1
2/1
RATIOS OF 3:
4/3
3/2
RATIOS OF 5:
6/5
5/4
8/5
5/3
RATIOS OF 7:
8/7 7/6
7/5 10/7
12/7 7/4
RATIOS OF 9:
10/9 9/8
9/7
14/9
16/9 9/5
RATIOS OF 11:
12/11 11/10
11/9
14/11
11/8
16/11
11/7
18/11
20/11 11/6
RATIOS OF 13:
14/13 13/12
13/11
16/13
13/10
18/13
13/9
20/13
13/8
22/13
24/13 13/7
RATIOS OF 15: (16/15) (15/14)
15/13
15/11
sqrt(2)
22/15
26/15
(28/15) (15/8)

In this paper we will consider two possible sets of consonant intervals as generators of scales and tuning systems. First, the 5-limit (consisting of the one Ratio of 3 and the two Ratios of 5). And then, the 7-limit (consisting of the one Ratio of 3, the two Ratios of 5, and the three Ratios of 7). The basic structures of each are shown in FIGURE 3.

First, let's examine the 5-limit. One can combine three notes so that each note is consonant with each of the other two notes, hence forming a closed triangle. There are two distinct possible structures that result: the major triad (top left of FIGURE 3) and the minor triad (top center of FIGURE 3). Each triad contains exactly one instance of every 5-limit consonant interval. These will be considered the two consonant 5-limit triads. The three pitches in each triad are given ratios so that 1/1 corresponds with the traditional and psychoacoustical[8] root of the triad, namely the pitch representing 2 in the 3:2 interval, rather than with Partch's Numerary Nexus. It is not possible to combine four or more pitches so that every pair forms a 5-limit consonance.

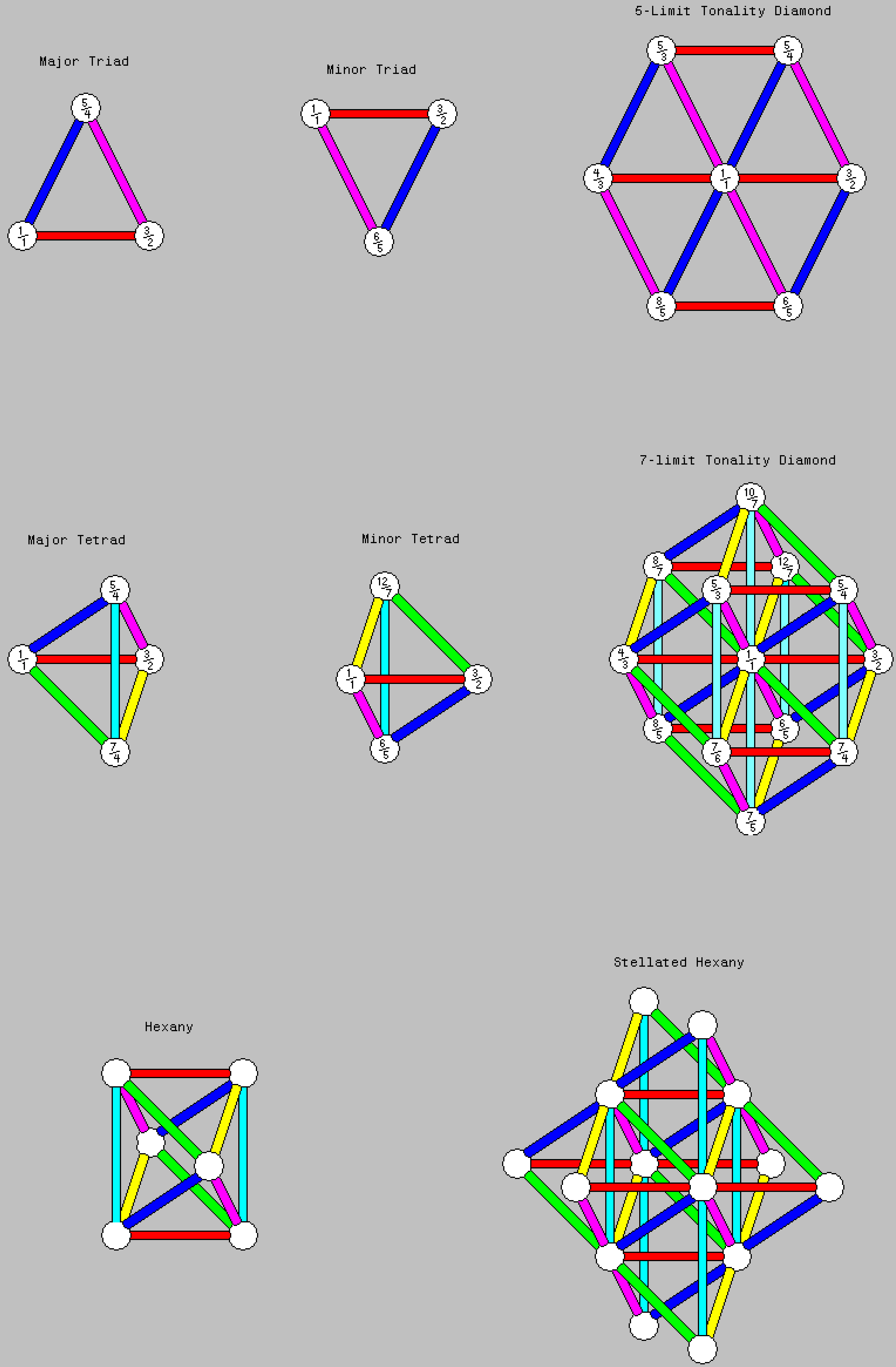
The 5-limit Tonality Diamond (top right of FIGURE 3) is a seven-pitch set that can be thought of in two ways. One way is as a central "1/1" pitch plus the set of all pitches lying at a 5-limit interval from it. The second way is as the union of all consonant 5-limit triads containing a central "1/1" pitch as a member. The 5-limit Tonality Diamond contains exactly four instances of each 5-limit consonant interval.

As will be seen, the full infinite lattice of pitches generated by combining 5-limit intervals can be seen as a two-dimensional plane divided equally into major and minor triads, with each pitch lying at the center of its own Tonality Diamond.

Now let's look at the 7-limit. The intervals are shown so as to suggest an orientation in three-dimensional space. One can combine four notes so that each note is consonant with each of the other three notes, hence forming a closed tetrahedron. There are two ways of doing this: the major tetrad (center left of FIGURE 3) and the minor tetrad (center of FIGURE 3). Each tetrad contains exactly one instance of every 7-limit consonant interval. These will be considered the two consonant 7-limit tetrads. The pitches in

[8] Parncutt, Richard. *Harmony: A Psychoacoustical Approach*. Springer Series in Information Sciences vol. 19, Springer-Verlag, Heidelberg, 1990, p. 70.

FIGURE 3


Major Triad
5/4
1/1
3/2
Minor Triad
1/1
3/2
6/5
5-Limit Tonality Diamond
5/3
5/4
4/3
1/1
3/2
8/5
6/5
Major Tetrad
5/4
1/1
3/2
7/4
Minor Tetrad
12/7
1/1
3/2
6/5
7-limit Tonality Diamond
10/7
8/7
12/7
5/3
5/4
4/3
1/1
3/2
8/5
6/5
7/6
7/4
7/5
Hexany
Stellated Hexany

each tetrad are given ratios so that 1/1 corresponds again with the pitch representing 2 in the 3:2 interval, rather than with Partch's Numerary Nexus.

It is not possible to combine five or more pitches so that every pair forms a 7-limit consonance. One can do so with three pitches, but the result will merely be a subset of one of the consonant tetrads.

The 7-limit Tonality Diamond (center right of FIGURE 3) is a thirteen-pitch set that can be thought of in two ways. One way is as a central "1/1" pitch plus the set of all pitches lying at a 7-limit interval from it. The second way is as the union of all consonant 7-limit tetrads containing a central "1/1" pitch as a member. The 7-limit Tonality Diamond contains exactly six instances of every 7-limit consonant interval.

As will be seen, the full infinite lattice of pitches generated by combining 7-limit intervals can be seen as a three-dimensional space, and each pitch will lie at the center of its own 7-limit Tonality Diamond. However, not all of the space will be contained within major and minor tetrads, as tetrahedra do not tile space. There will be octahedral gaps between the tetrads.

Each of these octahedra comprises a six-note set called the Hexany[9] (bottom left of FIGURE 3). The reader is free to arbitrarily choose one note of the Hexany as the 1/1 and fill in the other five ratios accordingly. The Hexany contains exactly two instances of each 7-limit consonant interval. A Hexany, together with the eight tetrads that border on it, collectively form a fourteen-pitch set known as a Stellated Hexany[10] (bottom right of FIGURE 3). The Stellated Hexany, like the 7-limit Tonality Diamond, contains six instances of each 7-limit consonant interval. Again, pitch ratios are left up to the reader.

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[9] Wilson, Ervin M. "D'Alessandro: Like a Hurricane", *Xenharmonikôn* vol. 12, 1989, pp. 1-39.
[10] Ibid.

The Diatonic Scale

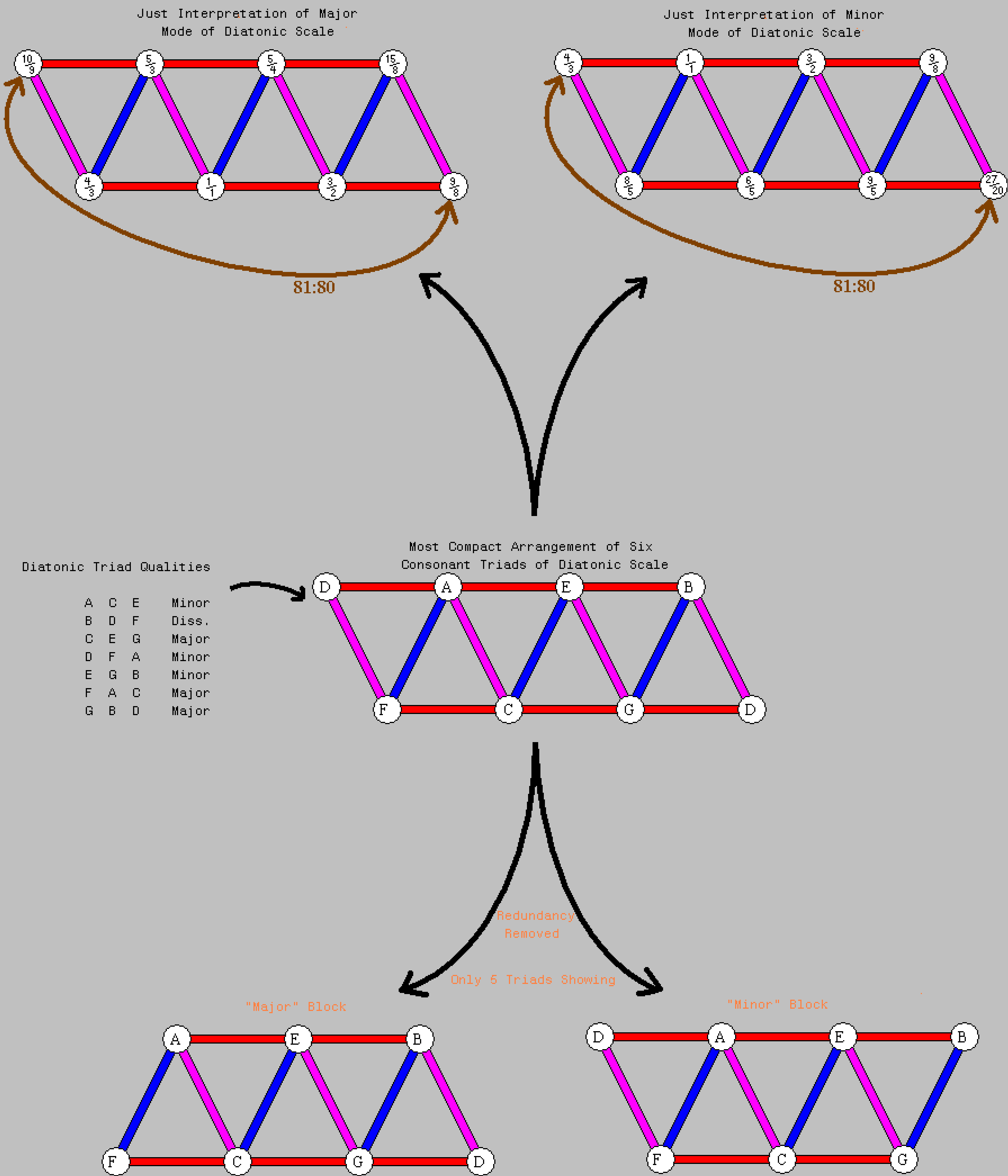
The diatonic scale is the basis of Western musical notation and most of its composition. At a particular pitch level, the diatonic scale is notated using the first 7 letters of the alphabet: A, B, C, D, E, F, and G. As shown in the table on the center-left of FIGURE 4, the pattern Root-3rd-5th is ordinarily assumed to result in consonant 5-limit triads in six out of its seven rotations through the diatonic scale. Unfortunately, it is not possible to represent these as interlocking triangles in the 5-limit lattice with each scale note in only one position. The most compact arrangement of the six triangles, shown in the center of FIGURE 4, is seen to require the note D to appear in two places in the lattice. One can assign pitch ratios to the notes in this arrangement by choosing one note as the 1/1 and filling in the other ratios accordingly. Two ways of doing this are shown – one corresponding to the major mode (C = 1/1, shown on the upper left of FIGURE 4), and one corresponding to the minor mode (A = 1/1, shown on the upper right of FIGURE 4). Either way, the two instances of the note D are represented by a pair of ratios that differ by the interval 80:81. This interval is known as the syntonic comma and is well known in Just Intonation investigations of the diatonic scale and diatonic music.

For our purposes of investigating periodicity, it will suffice to arbitrarily leave one of the D's (and hence one of the consonant triads) off the lattice arrangement, so that each scale note only appears in one place in the lattice. This results in one of the two lattice "blocks" on the bottom of FIGURE 4, which are typically associated in JI theory with the major and minor modes, respectively.

From the practical point of view of actually intoning a piece of music, leaving off one of the D's would be wholly unsatisfactory, as it would convert one of the consonant triads into a violently discordant sonority. There are at least five better methods of dealing with the two occurrences of D in the lattice, when a progression requires that a given notated D would be interpreted first in one sense and then immediately in the other:[11]

[11] Assuming the harmony is purely triadic. If more complex harmonies are used, further difficulties obtain with all the solutions except (3) and (5).

FIGURE 4


Just Interpretation of Major
Mode of Diatonic Scale
10/9 5/3 5/4 15/8
4/3 1/1 3/2 9/8
81:80
Just Interpretation of Minor
Mode of Diatonic Scale
4/3 1/1 3/2 9/8
8/5 6/5 9/5 27/20
81:80
Diatonic Triad Qualities
A C E Minor
B D F Diss.
C E G Major
D F A Minor
E G B Minor
F A C Major
G B D Major
Most Compact Arrangement of Six
Consonant Triads of Diatonic Scale
D A E B
F C G D
Redundancy
Removed
Only 5 Triads Showing
"Major" Block
A E B
F C G D
"Minor" Block
D A E B
F C G

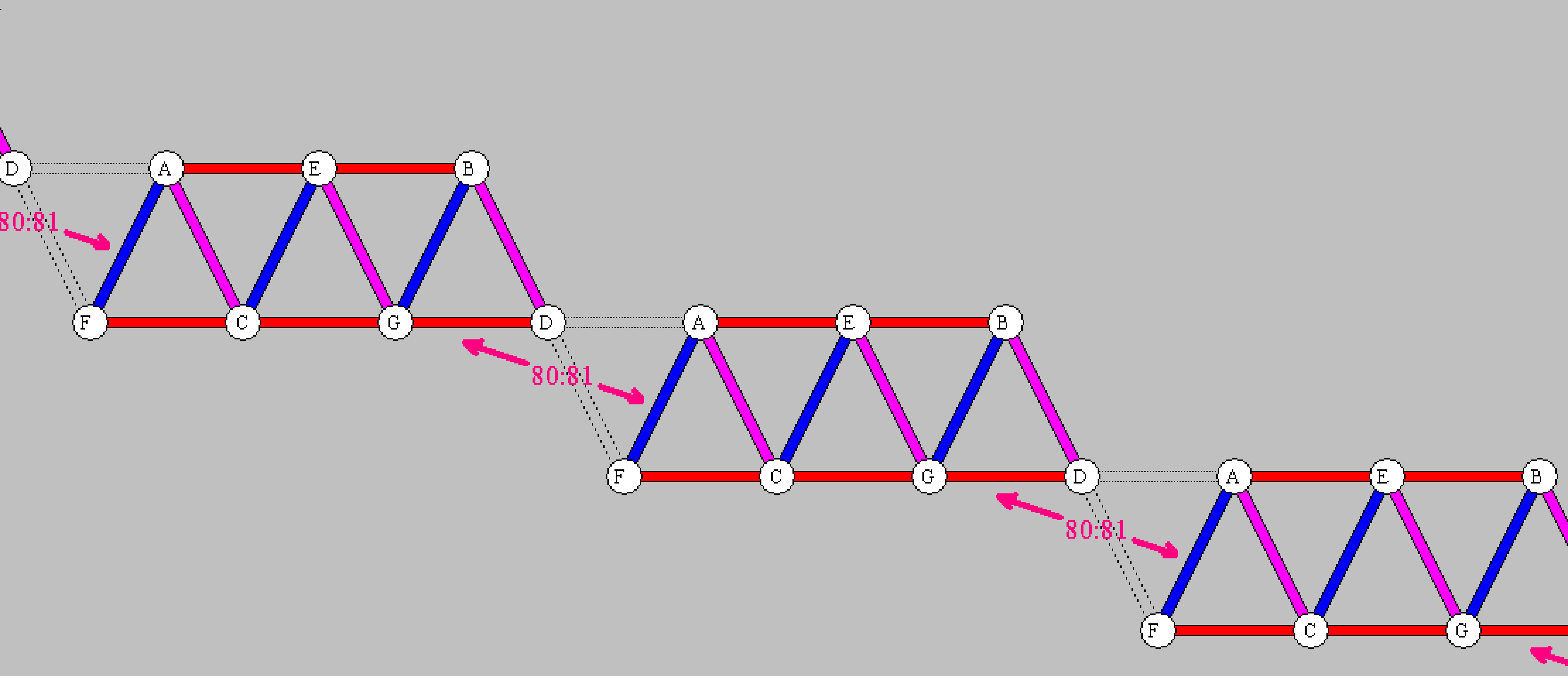
1) Strict JI: Use two different pitches for D, so that either may be used as required. (Disadvantage -- the full syntonic comma is a small but perceptible melodic shift; such a shift can disturb melodic-motivic coherence.)
2) Free-style JI: Instead of shifting the pitch of D in one direction, adjust the overall pitch level of the scale in the opposite direction so that there is no melodic shift in D. (Disadvantages –- many pieces of music would result in a drastic overall drift in pitch level; many pitches required for each scale note.)
3) Meantone temperament: Temper some of the consonant intervals by a fraction of a syntonic comma so that the two D's come out to the same pitch. (Disadvantage –- harmonic purity is lost.)
4) Adaptive JI:[12] Temper the melodic occurences of some of the consonant intervals by a fraction of a syntonic comma but keep the harmonies pure. (Disadvantage –- two slightly different[13] pitches are needed for each scale note.)
5) Adaptive tuning:[14] Define the "pain" of a rendition of a piece of music as some weighted combination of contributions from harmonic impurity, melodic shift, and overall pitch drift; then calculate the moment-to-moment pitch adjustments to minimize total "pain." (Disadvantage – a potentially unlimited number of slightly different pitches are needed for each scale note.)

Diatonic Periodicity

Having arbitrarily chosen the "major block" to depict the diatonic scale, let's show the occurrences of this block in the infinite 5-limit lattice. Look first at FIGURE 5. This is the band of the 5-limit lattice that corresponds notationally to the unaltered diatonic scale. You will see several repetitions of the "major block", which are transpositions of one another by the interval 80:81. Again, the syntonic comma 80:81 is not distinguished in Western musical notation and so the notation in the infinite 5-limit lattice simply repeats itself at every 80:81 increment. The lattice could represent an

[12] First described in Vicentino, Nicola. *L'antica musica ridotta alla moderna prattica*. Antonio Barre, Rome, 1555, 1557. English translation *Ancient music adapted to modern practice* by Maria Rika Maniates (ed.), Claude V. Palisca (ed.), Yale University Press, New Haven CT, 1996, p. xlix.
[13] In Vicentino's system, they differ by about ¼ syntonic comma, or about 5½ cents.
[14] John deLaubenfels has made the most advanced forays into this area: http://www.adaptune.com/.

FIGURE 5



actual set of distinct pitches that are traversed in the course of a free-style JI rendition of a diatonic piece, with 80:81 considered too small an interval to warrant a change in notation. Or it could depict the repetition of pitches that would occur if the same piece were played in meantone temperament or adaptive JI, where 80:81 gets tempered out of existence. In any case, observe that every pitch in the lattice belongs to one and only one "major block." It shouldn't be difficult to convince yourself that the same properties would obtain had we used the "minor block" instead.

Now move on to FIGURE 6. Here a much wider portion of the 5-limit lattice is shown, corresponding to what is accessible using the standard accidentals ranging from double flats to double sharps. First, observe that by moving by a consonant interval away from the unaltered diatonic block, we obtain a sharp or flat note, usually a 25:24 away from the unaltered note with the same letter name within the original block, occasionally a 128:135 away. (In particular, moving up a 25:24 or a 135:128 adds a sharp to, or removes a flat from, the notation of a note; while moving down a 24:25 or a 128:135 adds a flat to, or removes a sharp from, the notation of a note.) For this reason, 25:24 is usually referred to as the "chromatic semitone" or "augmented unison" in JI theory, while 128:135 (which is simply 80:81 * 24:25) goes by the fancier names of "limma ascendant,"[15] "major limma," and "large chroma".[16] Whatever the particulars of the tuning, these sharpening/flattening intervals are larger than the notationally ignored syntonic comma –- yet they are smaller than the smallest interval in the scale itself (in JI, a 16:15 or "diatonic semitone") -- hence the use of special sharp and flat signs is logical.

Next, observe that in this way, the entirety of the infinite 5-limit lattice can be divided into identical "major blocks". Each block has exactly one of each letter name, and the same chromatic modifier applied to all seven notes in the block. The blocks are separated in the lattice by intervals of 80:81, 25:24, and 128:135. Every note in the lattice belongs to one and only one of these blocks. Technically, we say that the diatonic scale is the 5-limit periodicity block[17] defined by the pair of unison vectors 80:81 and 25:24 (or 80:81 and 128:135). Notationally, it is evident that 80:81 serves as a

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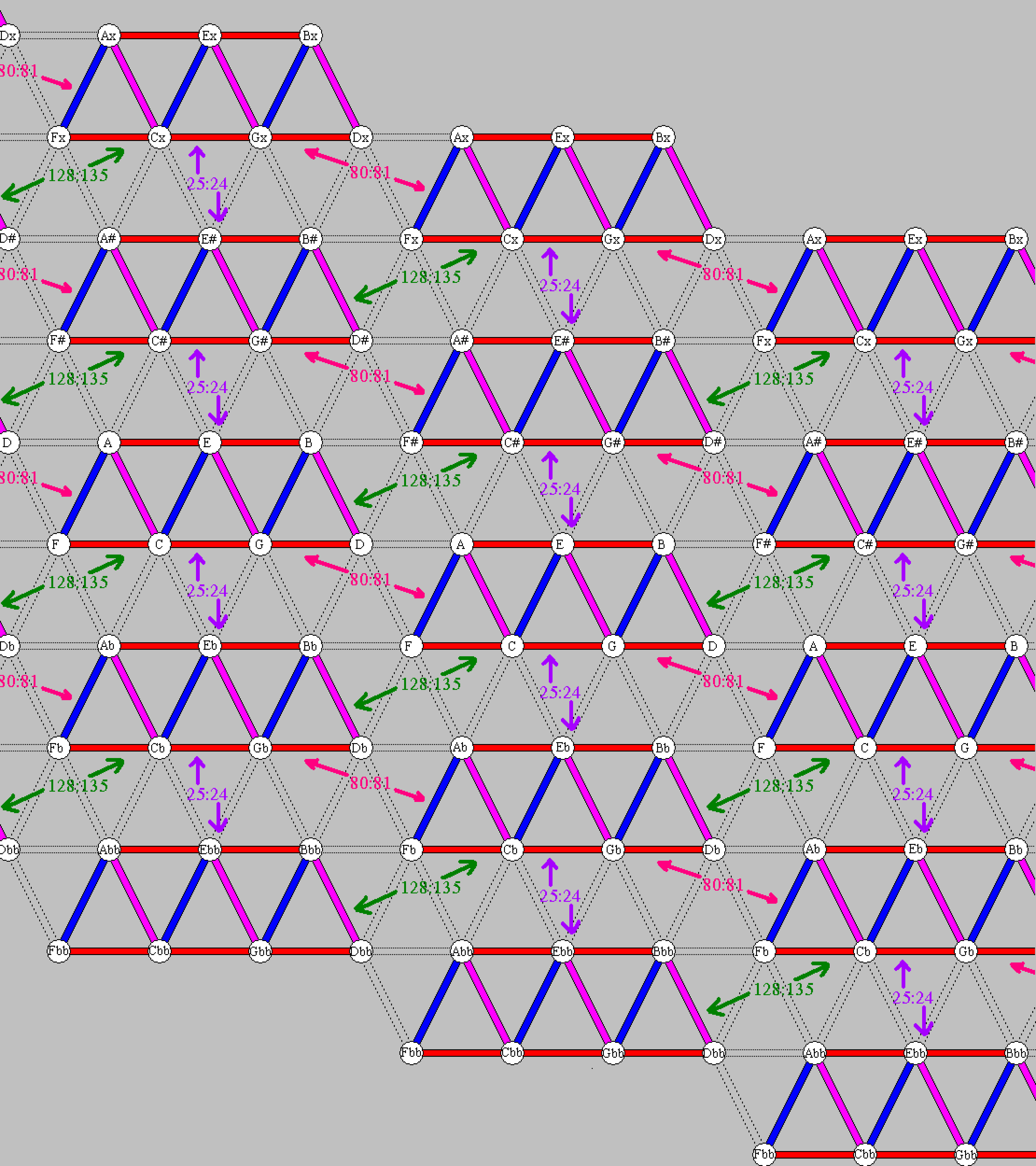
[15] Gann, Kyle, *Anatomy of an Octave.* http://home.earthlink.net/~kgann/Octave.html.

[16] Op de Coul, Manuel, http://www.xs4all.nl/~huygensf/doc/intervals.html.

[17] Erlich, Paul. *A Gentle Introduction to Fokker Periodicity Blocks.* http://sonic-arts.org/td/erlich/intropblock1.htm.

FIGURE 6



Dx Ax Ex Bx
80:81
Fx Cx Gx Dx Ax Ex Bx
128:135 25:24 80:81
D# A# E# B# Fx Cx Gx Dx Ax Ex Bx
80:81 128:135 25:24 80:81
F# C# G# D# A# E# B# Fx Cx Gx
128:135 25:24 80:81 128:135 25:24
D A E B F# C# G# D# A# E# B#
80:81 128:135 25:24 80:81
F C G D A E B F# C# G#
128:135 25:24 80:81 128:135 25:24
Db Ab Eb Bb F C G D A E B
80:81 128:135 25:24 80:81
Fb Cb Gb Db Ab Eb Bb F C G
128:135 25:24 80:81 128:135 25:24
Dbb Abb Ebb Bbb Fb Cb Gb Db Ab Eb Bb
128:135 25:24 80:81
Fbb Cbb Gbb Dbb Abb Ebb Bbb Fb Cb Gb
128:135 25:24
Fbb Cbb Gbb Dbb Abb Ebb Bbb
Fbb Cbb Gbb

commatic unison vector, while 25:24 or 128:135 serves as a *chromatic* unison vector.[18] And again, it should be evident that the "minor block" would have served just as well for this illustration.

Finally, let's verify from an accounting perspective that the entire 5-limit lattice is covered by these periodicity blocks. First, note that the diatonic scale has 7 notes. Since each note in the lattice is the center of its own 5-limit Tonality Diamond, it is consonant with 6 other notes in the lattice. Now since each consonance involves 2 notes, each unit of periodicity in the lattice must contain 7*6/2 = 21 consonant intervals. In FIGURE 6, 11 of these are colored. 6 of them are involved in connecting the block to its duplicate at 25:24 to the north (the connectors to the block a 24:25 to the south are simply duplicates of these 6). 2 of them are involved in connecting the block to its duplicate at 128:135 to the southwest. And 2 more connect the block to its duplicate at 80:81 to the northwest. Since 11 + 6 + 2 + 2 = 21, we have proved that moving from any note in one block by a 5-limit consonant interval will always lead to a note in the same or in a duplicate block. And since the 5-limit lattice is the complete network of 5-limit consonant intervals, it follows that every note in the lattice must be a member a diatonic block.

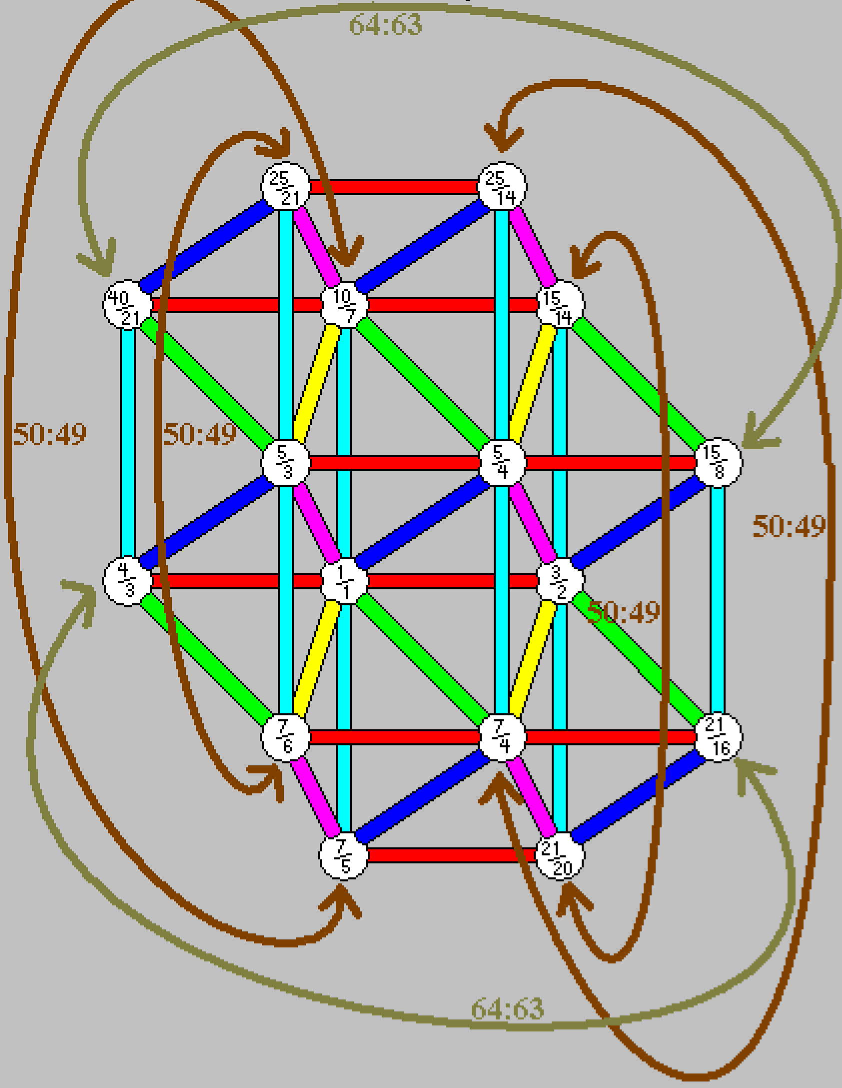
The Symmetrical Decatonic Scale

The author first introduced the decatonic scales in the context of 22-tone equal temperament and similar tunings,[19] but at least one theorist[20] has noted a natural 10-tone periodicity in the 7-limit JI lattice. The author notated the symmetrical decatonic scale using the 10 arabic numerals: '1', '2', '3', '4', '5', '6', '7', '8', '9', and '0'. The table on the center-left of FIGURE 7 shows how the pattern Root-4th-7th-9th (decatonically speaking -- the "octave" would be called an 11th) results in consonant 7-limit triads in eight out of its ten rotations through the symmetrical decatonic scale (this should clarify why the moniker "symmetrical" is used). Unfortunately, it is not possible to represent these as interlocking tetrahedra in the 7-limit lattice with each scale note in only one position. The most compact arrangement of

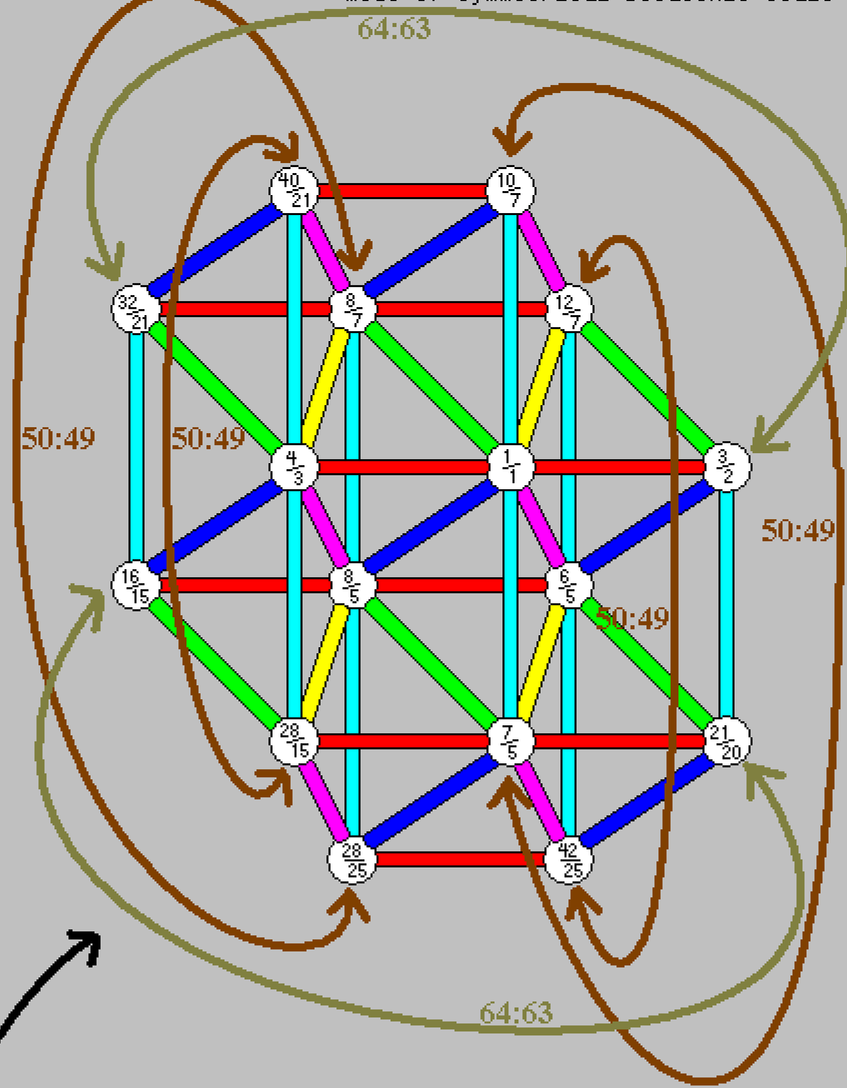
[18] I am indebted to Paul Hahn for introducing this terminology,
[19] Erlich, Paul. "Tuning, Tonality and Twenty-Two-Tone Temperament", *Xenharmonikôn* vol. 17, spring 1998, pp. 12-40.
[20] Prooijen, Kees van. *Searching Small Intervals*. http://www.kees.cc/tuning/perbl.html.

FIGURE 7

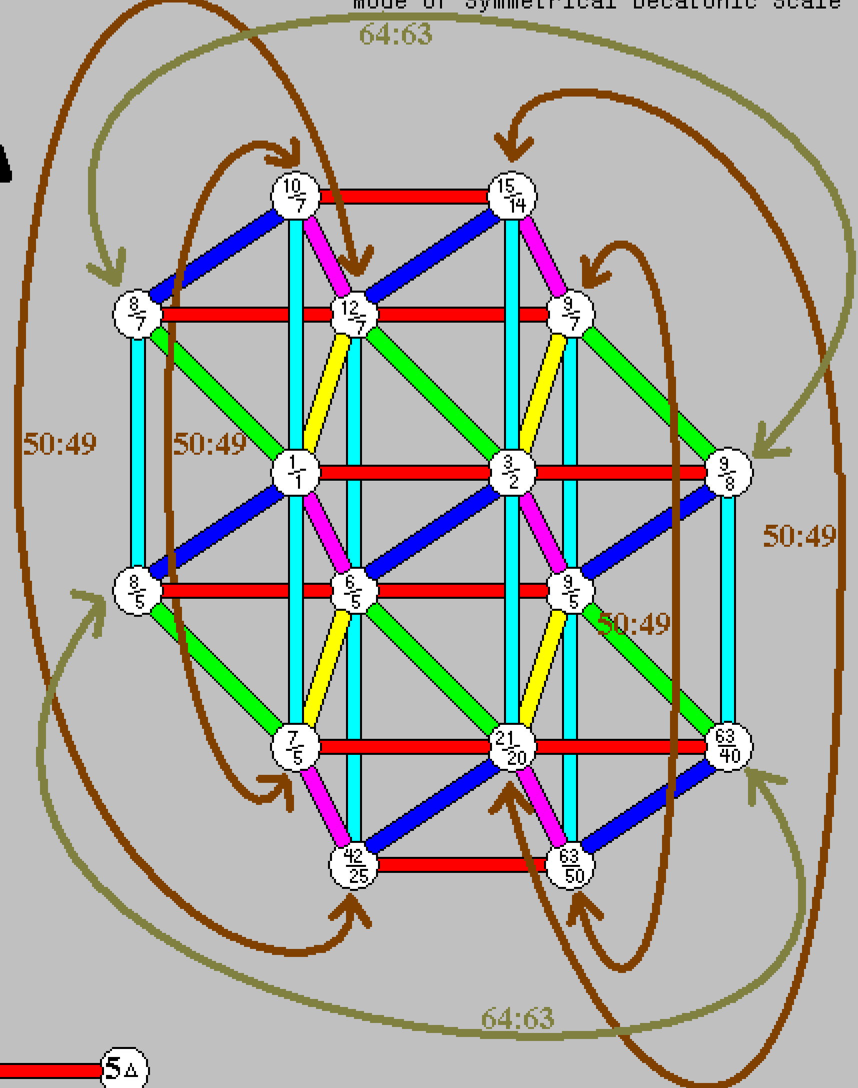
Just Interpretation of Dynamic Major Mode of Symmetrical Decatonic Scale



Just Interpretation of Static Minor Mode of Symmetrical Decatonic Scale



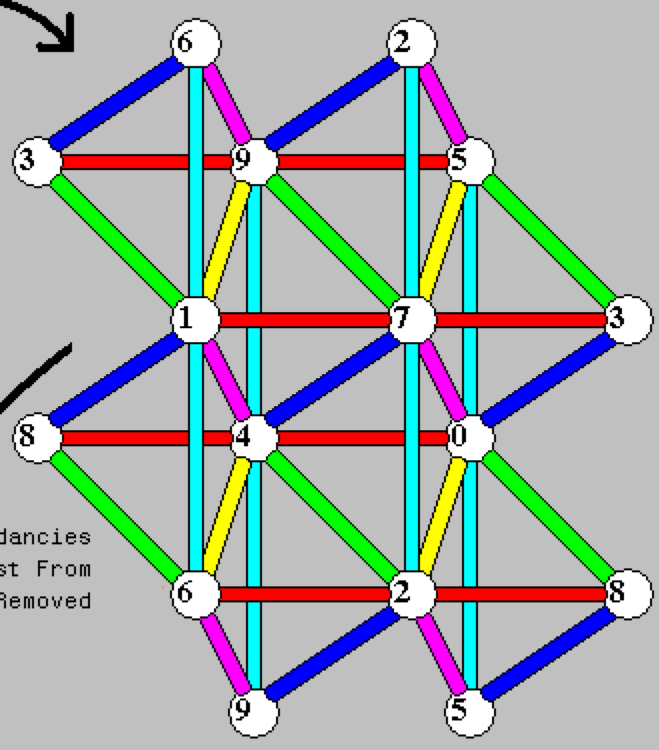
Just Interpretation of Dynamic MInor Mode of Symmetrical Decatonic Scale



Symmetrical Decatonic Tetrad Qualities

1	4	7	9	Minor
2	5	8	0	Minor
3	6	9	1	Major
4	7	0	2	Major
5	8	1	3	Diss.
6	9	2	4	Minor
7	0	3	5	Minor
8	1	4	6	Major
9	2	5	7	Major
0	3	6	8	Diss.

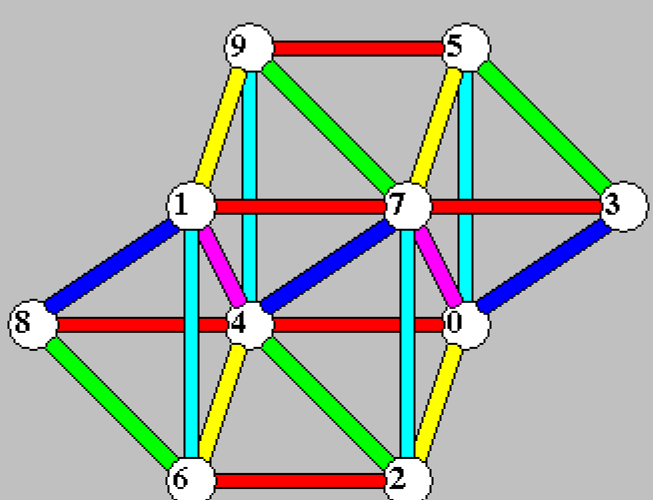
Most Compact Arrangement of Eight Consonant Tetrads of Symmetrical Decatonic Scale



Redundancies Furthest From Center Removed

Only 4 Tetrads Showing

"Symmetrical Decatonic" Block



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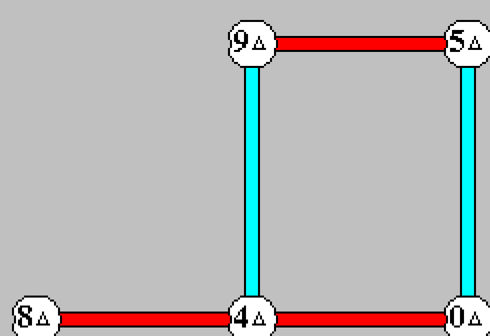
Back Layer



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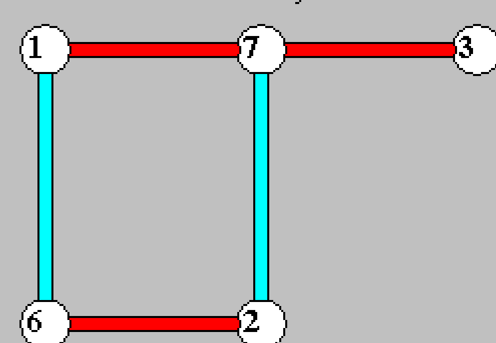
Front Layer



the eight tetrahedra, shown in the center of FIGURE 7, is seen to require the notes '2', '3', '5', '6', '8', and '9' each to appear in two places in the lattice.

One can assign pitch ratios to the notes in this arrangement by choosing one note as the 1/1 and filling in the other ratios accordingly. Three ways of doing this are shown. One corresponds to the Dynamic Major mode ('4' = 1/1, shown on the upper left of FIGURE 7). One corresponds to the Static Minor mode ('7' = 1/1, shown on the upper right of FIGURE 7). And one corresponds to the Dynamic Minor mode ('1' = 1/1, shown on the center right of FIGURE 7). In all cases, the two instances of the note '2', of the note '5', of the note '6', and of the note '9' are represented by pairs of ratios that differ by the interval 50:49. Additionally, the two instances of the note '3' and of the note '8' are represented by pairs of ratios that differ by the interval 64:63. Evidently, these are the commatic unison vectors associated with the symmetrical decatonic scale.

For our purposes of investigating periodicity, we will leave these redundancies off the lattice arrangement, so that each scale note only appears in one place in the lattice. We choose the member of each pair which appears closer to the center of the lattice diagram.[21] This results in the lattice "block" on the bottom left of FIGURE 7. To aid in seeing the three-dimensional structure of this block, it is shown as the union of two two-dimensional subsets –- a "front layer" and a "back layer". These subsets have no special musical significance, and a different orientation of the lattice could result in a different breakdown along these lines.

Decatonic Periodicity

Having chosen our "symmetrical decatonic block", let's show the occurrences of this block in the infinite 7-limit lattice. Look first at FIGURE 8. This is the "slice" through the 7-limit lattice that corresponds notationally to the unaltered decatonic scale. You will see several repetitions of the "symmetrical decatonic block", which are transpositions of one another by the intervals 50:49, 64:63, and 225:224 (which is simply 50:49 * 63:64). Again, the commas 50:49 and 64:63 are not distinguished in decatonic

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[21] This choice is made in the interest of *compactness*, which could be defined as maximizing the number of consonant relationships for a given number of notes.

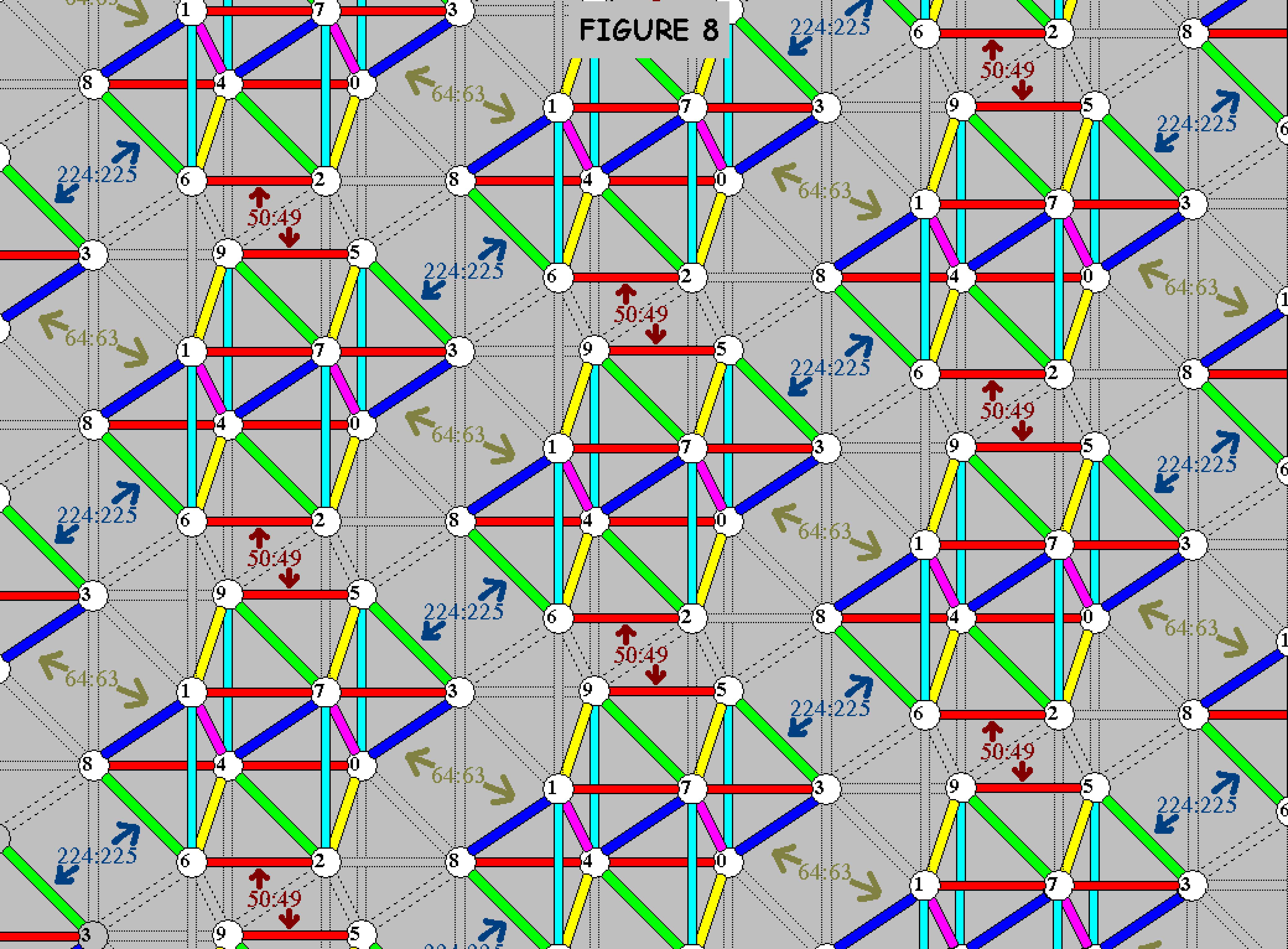


FIGURE 8

musical notation and so the notation in the infinite 7-limit lattice simply repeats itself at every increment of 50:49, 64:63, and 225:224. The lattice could represent an actual set of distinct pitches that are traversed in the course of a free-style JI rendition of a decatonic piece, with 50:49 and 64:63 considered too small an interval to warrant a change in notation. Or it could depict the repetition of pitches that would occur if the same piece were played in "paultone" temperament[22] or adaptive JI, where 50:49, 64:63, and 225:224 get tempered out of existence. In any case, observe that every pitch in the lattice belongs to one and only one "symmetrical decatonic block". Finally, we count that two blocks a 50:49 apart connect through 7 consonant intervals; two blocks a 64:63 apart connect through 5 consonant intervals; and two blocks a 225:224 apart connect through 3 consonant intervals.

Now move on to FIGURE 9. This is a reproduction of the blocks in FIGURE 8 with one new block introduced. One sees small upward-triangles modifying the note names in the new block; analogous to sharps, these represent upward alterations in the pitch of the basic decatonic notes. Observe that the new block results from transposing one of the old blocks up by either a 25:24, a 28:27, or a 49:48. These are all larger than the notationally ignored commas, yet smaller than the smallest decatonic step (in JI, a 21:20) -- hence, the use of a special "up" sign is logical. The number of consonant intervals involved in connecting the new block to it neighbor 24:25 below is 5; connecting it to its neighbor 27:28 below are 6 consonant intervals, and to its neighbor 48:49 below, 11 consonant intervals.

FIGURE 10 is a large chunk of the 7-limit lattice, four layers deep. Logically, in addition to the "up" signs, one sees "down" signs (triangles pointing downward), as well as a few "double ups" and "double downs". One can see many copies of the "symmetrical decatonic block", as well as the single-layer sections of it (cf. the bottom of FIGURE 7) that result from FIGURE 10 being only a finite number of layers deep. Though it may not be immediately obvious from looking at FIGURE 10, the periodicity of the decatonic block does completely account for the three-dimensional 7-limit lattice, much as the diatonic block does for the two-dimensional 5-limit lattice. Since we've been counting consonant intervals, though, we are in a position to verify this numerically. The decatonic scale has, of course, 10

[22] This refers to a temperament consisting of two chains of 707¢ to 711¢ "fifths", a half-octave apart from one another. 22-tone equal temperament is an example.

FIGURE 9

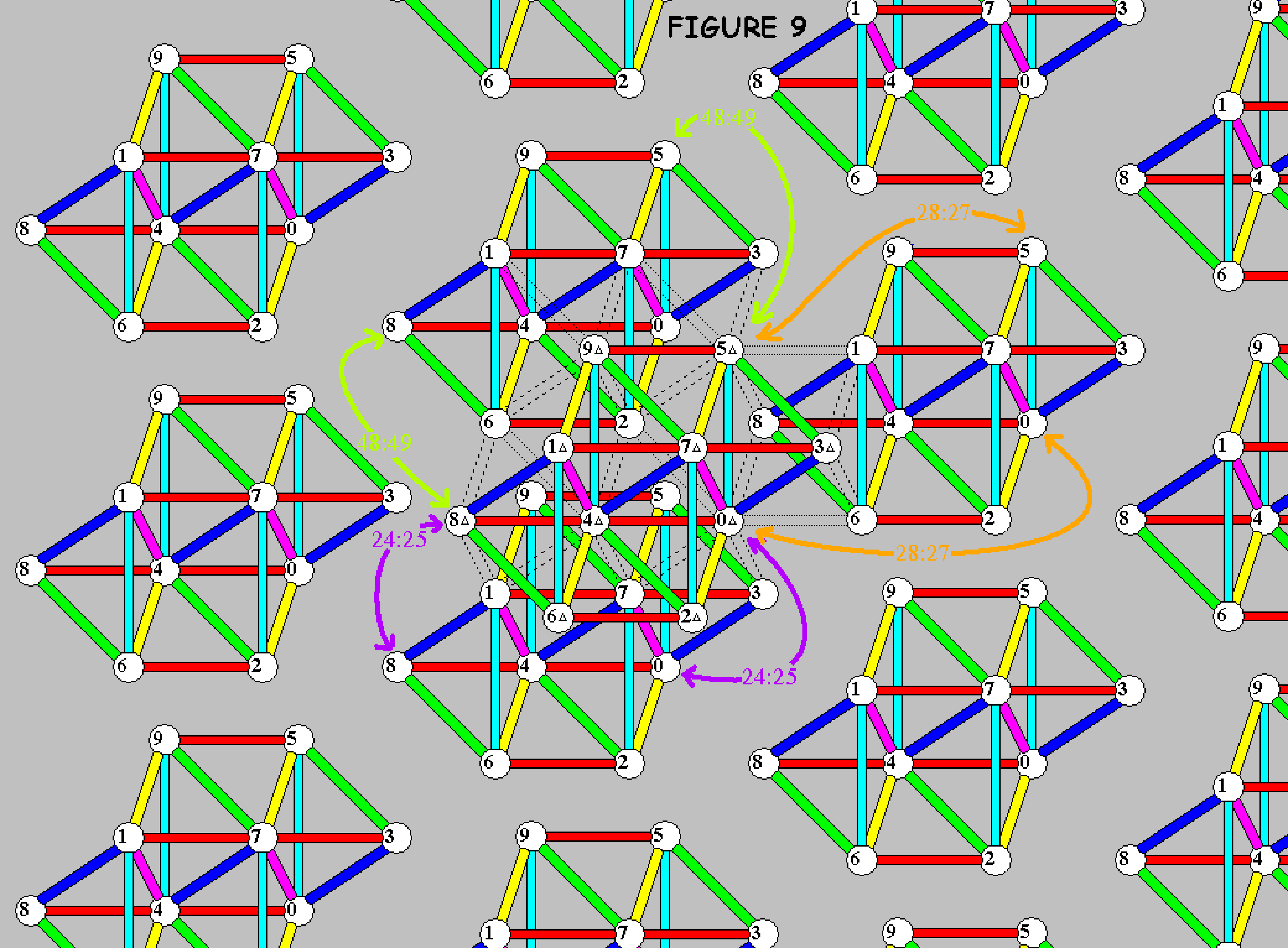
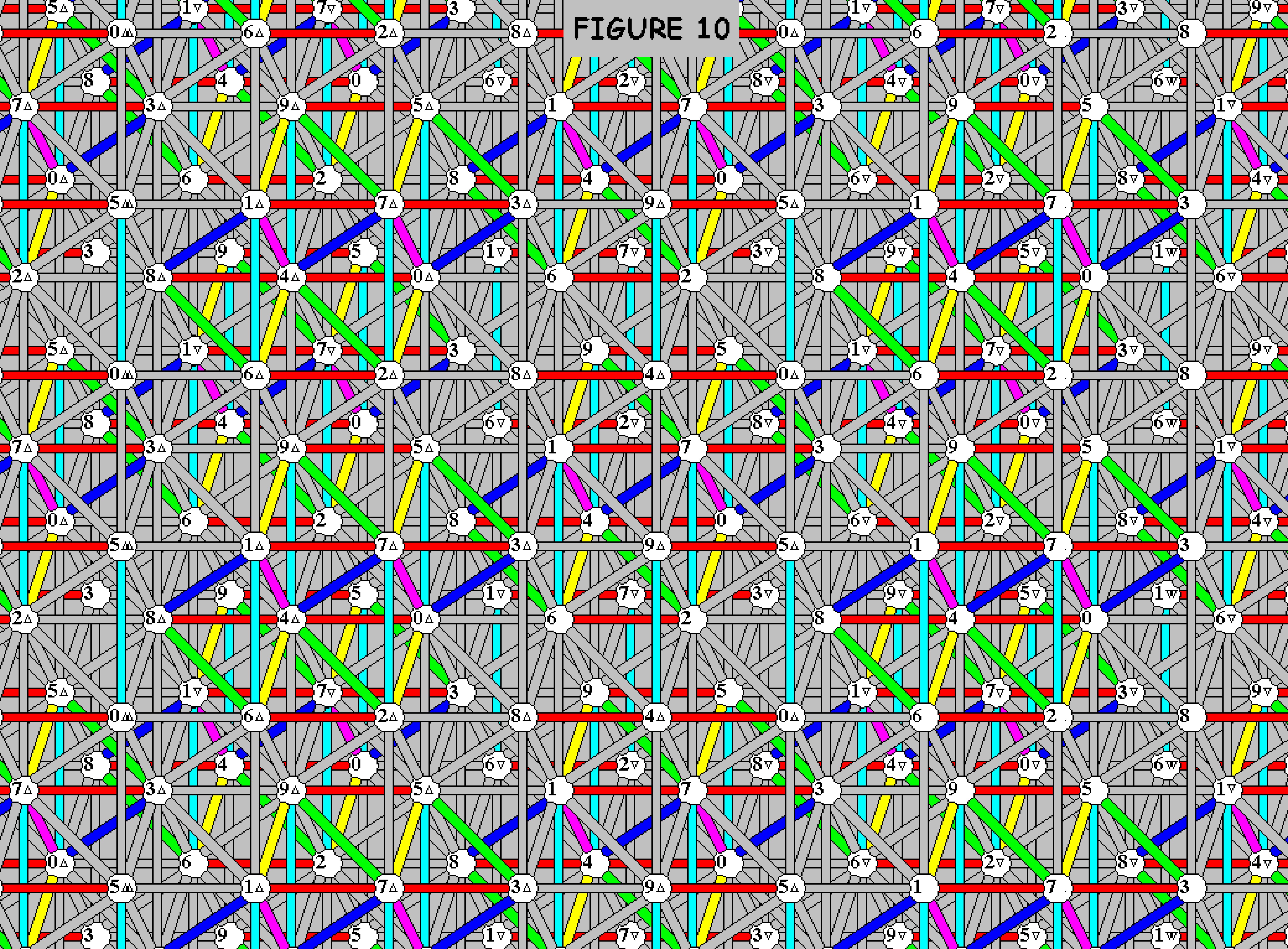


FIGURE 10



notes. Since each note in the lattice is the center of its own 7-limit Tonality Diamond, it is consonant with 12 other notes in the lattice. Now since each consonance involves 2 notes, each unit of periodicity in the lattice must contain 10*12/2 = 60 consonant intervals. The symmetrical decatonic block itself comprises 23 colored lines. We've counted all the other consonant intervals, those connecting one block to another, and now we simply add -- 7 + 5 + 3 = 15 consonant intervals between blocks with identical nomenclature, and 5 + 6 + 11 = 22 consonant intervals between blocks differing by an "up" or a "down". And since 23 + 37 = 60, every consonant interval in the lattice must be either connecting notes within a block or connecting notes among duplicate blocks.

On the last page of this paper I've included a design (by Steve Rezsutek) of a 22-tone keyboard in which the decatonic scale is mapped to the black keys and the altered notes to the white keys. Also shown is a table of decatonic key signatures,[23] including signatures for the "pentachordal decatonic" scales, not discussed here, but possessing melodic qualities similar to those the diatonic scale obtains from its pairs of identical tetrachords.

Additional References

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•Vogel, Martin. *On the Relations of Tone*, translated from German by Vincent Jean Kisselbach and edited by Carl A. Poldy, Verlag für systematische Musikwissenschaft, Bonn-Bad Godesberg, 1993.

[23] Erlich, op. cit. The table of key signatures there is incorrect – this is a corrected version.

Decatonic Key Signatures and Rezsutek Keyboard Layout

The key signatures in the first and third columns refer to the standard pentachordal major and minor modes built on the indicated tonics. For the alternate pentachordal major and minor modes, match the key signatures in the first column with the respective tonics in the third column, and vice versa.