Tuning Continua and Keyboard Layouts

by William Sethares, Andrew Milne and Jim Plamondon

A continuously parameterized family of tunings can be mapped to a button field so that the geometric shape of each musical interval is the same across all keys and throughout all tunings in the continuum.
Transpositional Invariance

In this layout, intervals and chords are fingered the same in all keys.
Idea of Tuning Invariance

Can we do the same kind of thing (have consistent fingering) across a range of tunings, instead of across all keys in a single tuning?

For example, can we arrange things so that (say) a 12-tet major chord, a 17-tet major chord, and a Pythagorean major chord all have the same fingering (while retaining transpositional invariance)?

When possible, there are several advantages:

- ease of learning new tunings
- ease of visualizing underlying structure of the music
- possibility of dynamically (re)tuning all sounded notes in real time throughout various tunings
Two Mappings and an Issue:

There are two mappings involved in the process: the first tempers from an arbitrary regular tuning to one that can be represented by a finite number of generators.

The second mapping is from the generators to the button field: translation invariance is shown to be equivalent to the linearity of this mapping, and consistent fingering occurs when the linear mapping is also invertible.

Issue: what does it mean to be the “same interval” or the “same chord” in multiple tunings?
Two Mappings:

Regular Temperament

Just Intonation tuning system

Temperament Mapping

characterised by null space of comma(s)

Lower Rank Tuning System

Layout Mapping

linear and invertible

Layout

n-dimensional Button Lattice
Tempering by Commas: the General Case

Suppose a system $S$ contains $p$ generators $g_1, g_2, \ldots, g_p$ where any element $s \in S$ can be expressed as $g_1^{i_1} g_2^{i_2} \cdots g_p^{i_p}$ for integers $i_j$. The generators are *tempered* by $n < p$ commas, which means that the basis elements are replaced by nearby values

$$g_1 \rightarrow G_1, g_2 \rightarrow G_1, \ldots, g_p \rightarrow G_p$$

where

$$G_1^{c_1^{11}} G_2^{c_1^{12}} \cdots G_p^{c_1^{1p}} = 1$$
$$G_1^{c_2^{21}} G_2^{c_2^{22}} \cdots G_p^{c_2^{2p}} = 1$$
$$\vdots$$
$$G_1^{c_n^{n1}} G_2^{c_n^{n2}} \cdots G_p^{c_n^{np}} = 1.$$
This set of constraints reduces the dimension from rank \( p \) to rank \( p - n = r \). Gather the coefficients of the commas into the matrix

\[
C = \begin{pmatrix}
c_{11} & c_{12} & \cdots & c_{1p} \\
c_{21} & c_{22} & \cdots & c_{2p} \\
\vdots & \vdots & \ddots & \vdots \\
c_{n1} & c_{n2} & \cdots & c_{np}
\end{pmatrix}
\]

and let \( \mathcal{N}(C) \) be a basis for the null space of \( C \). Then the range space mapping \( R: \mathbb{Z}^p \rightarrow \mathbb{Z}^n \) has a basis defined by the transpose of \( \mathcal{N}(C) \).
Tempering by Commas: Example I

Consider the 5-limit primes defined by the generators 2, 3, 5, which are tempered to $2 \rightarrow G_1$, $3 \rightarrow G_2$, and $5 \rightarrow G_3$ by the syntonic comma $G_1^{-4}G_2^4G_3^{-1} = 1$ and the major diesis $G_1^3G_2^4G_3^{-4} = 1$. Then

$$C = \begin{pmatrix} -4 & 4 & -1 \\ 3 & 4 & -4 \end{pmatrix}$$

has null space $\mathcal{N}(C) = (12, 19, 28)'$. Thus $R = (12, 19, 28)$, and a typical element $2^{i_1}3^{i_2}5^{i_3}$ is tempered to $G_1^{i_1}G_2^{i_2}G_3^{i_3}$ and then mapped by $R$ to $12i_1 + 19i_2 + 28i_3$. All three temperings can be written in terms of a single variable $\alpha$ as $G_1 = \alpha^{12}$, $G_2 = \alpha^{19}$, and $G_3 = \alpha^{28}$. If the choice is made to temper $G_1$ to 2 (to leave the octave unchanged) then $\alpha = \frac{12\sqrt{2}}{2}$ and the result is 12-tone equal temperament.
The 5-limit primes defined by the generators 2, 3, 5, may be tempered to $2 \rightarrow G_1$, $3 \rightarrow G_2$, and $5 \rightarrow G_3$ by the syntonic comma $G_1^{-4}G_2^4G_3^{-1} = 1$. Then $C = (-4, 4, -1)$ has null space spanned by the rows of

$$R = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 4 \end{pmatrix}.$$ 

Typical elements $2^{i_1}3^{i_2}5^{i_3}$ are tempered to $G_1^{i_1}G_2^{i_2}G_3^{i_3}$ and then mapped via

$$R \begin{pmatrix} i_1 \\ i_2 \\ i_3 \end{pmatrix} = \begin{pmatrix} i_1 + i_2 \\ -i_1 + 4i_3 \end{pmatrix}.$$ 

The tempered generators can be written in terms of two basis elements $\alpha$ and $\beta$ of the columns of $R$ as $G_1 = \alpha\beta^{-1}$, $G_2 = \alpha$, and $G_3 = \beta^4$. 
\(\alpha\text{-Reduced } \beta\text{-Chains}\)

are generated by stacking integer powers of \(\beta\) and then \textit{reducing} (dividing or multiplying by \(\alpha\)) so that every term lies between 1 and \(\alpha\). For any \(i \in \mathbb{Z}\), the \(i\)th note is

\[ \beta^i \alpha - \lfloor i \log_\alpha(\beta) \rfloor \]

where \([x]\) represents the largest integer less than or equal to \(x\). \(\alpha\)-reduced \(\beta\)-chains define scales that repeat at intervals of \(\alpha\); \(\alpha = 2\), representing repetition at the octave, is the most common value.
Valid Tuning Range

Any interval $s \in S$ can be written in terms of the $p$ generators as $s = g_1^{i_1} g_2^{i_2} \cdots g_p^{i_p}$ where $i_j \in \mathbb{Z}$, or more concisely as the vector $s = (i_1, i_2, \cdots, i_p)$. A set of $n$ commas defines the temperament mapping $R$. Consider a privileged set of intervals

$$1 = s_0 < s_1 < s_2 < \ldots < s_Q,$$

which can also be represented as the vectors $s_0, s_1, \ldots, s_Q$. Given any set of generators $\alpha_1, \alpha_2, \ldots, \alpha_{p-n}$ for the reduced rank tuning system, each interval $s_i$ is tempered to $Rs_i = \alpha_1^{j_1} \alpha_2^{j_2} \cdots \alpha_{p-n}^{j_{p-n}}$ where $j_k \in \mathbb{Z}$. The $\alpha$-generators define the specific tempered tuning and the coefficients $j_k$ specify the exact ratios of the privileged intervals within the temperament. The set of all generators $\alpha_i$ for which

$$1 = Rs_0 < Rs_1 < Rs_2 < \ldots < Rs_Q$$

holds is called the valid tuning range (VTR).
Example: The Primary Consonances

Perhaps the most common example of a privileged set of intervals in 5-limit JI is the set of eight common practice consonant intervals

\[ 1, \frac{6}{5}, \frac{5}{4}, \frac{4}{3}, \frac{3}{2}, \frac{8}{5}, \frac{5}{3}, 2 \]

which are familiar to musicians as the unison, just major and minor thirds, just perfect fifth, octave, and their octave inversions.
Finding the VTR

The requirement that $R_{s_{i+1}} > R_{s_i}$ is identical to the requirement that $Sx > 0$ where

$$S = \begin{pmatrix} s_1 - s_0 \\ s_2 - s_1 \\ \vdots \\ s_{p-n} - s_{p-n-1} \end{pmatrix}$$

and

$$x = \begin{pmatrix} \log(\alpha_1) \\ \log(\alpha_2) \\ \vdots \\ \log(\alpha_{p-n}) \end{pmatrix}.$$  

(The inequality signifies an element-by-element operation.) This is the intersection of $p - n$ half-planes with boundaries that pass through the origin. The monotonicity assumption guarantees that the intersection is a nonempty cone radiating from the origin; this cone defines the VTR.
Example: 5-Limit Syntonic Continuum

The primary consonances are mapped by \( \mathbb{R} \) to

\[
\begin{pmatrix}
0 & 5 & -6 & 2 & -1 & 7 & -4 & 1 & 0
\end{pmatrix}^T
\]

In terms of the generators \( \alpha \) and \( \beta \) this is

\[
\alpha^0 \beta^0 < \alpha^5 \beta^{-3} < \alpha^{-6} \beta^4 < \alpha^2 \beta^{-1} < \alpha^{-1} \beta^1 < \alpha^7 \beta^{-4} < \alpha^{-4} \beta^3 < \alpha^1 \beta^0
\]

Rewriting this as a matrix gives

\[
\begin{pmatrix}
5 & -11 & 8 & -3 & 8 & -11 & 5 \\
-3 & 7 & -5 & 2 & -5 & 7 & -3
\end{pmatrix}^T
\begin{pmatrix}
x_1 \\ x_2
\end{pmatrix}
> \begin{pmatrix}
0 \\ 0
\end{pmatrix}.
\]

This region is the cone bounded below by \( x_2 = \frac{11}{7} x_1 \) and bounded above by \( x_2 = \frac{8}{5} x_1 \), or \( \alpha^\frac{11}{7} < \beta < \alpha^\frac{8}{5} \). For \( \alpha = 2 \), this covers the range between 7-edo and 5-edo. Outside this range, one or more of the privileged intervals changes fingering. This VTR range is identical to Blackwood’s range of recognisable diatonic tunings, to the 12-note MOS scale generated by fifths.
Valid Tuning Ranges: With $\alpha = 2$, the size of major second (M2), minor second (m2), and augmented unison (AU) over a range of $\beta$. 
The VTR for the syntonic continuum (given by the primary consonances) corresponds to Blackwood’s range of “recognizable diatonic tunings” and to the 12-note MOS scale generated by fifths.
VTRs are Easy:

A selection of temperaments with $2 \rightarrow \alpha$ and $1 < \beta < 2^{\frac{1}{2}}$. VTR values for the primary consonances are rounded to the nearest cent and the comma vectors presume that $2 \rightarrow G_1$, $3 \rightarrow G_2$, $5 \rightarrow G_3$.

<table>
<thead>
<tr>
<th>Common name</th>
<th>Negri</th>
<th>Porcupine</th>
<th>Hanson</th>
<th>Magic</th>
</tr>
</thead>
<tbody>
<tr>
<td>Comma</td>
<td>$(-14, 3, 4)$</td>
<td>$(1, -5, 3)$</td>
<td>$(-6, -5, 6)$</td>
<td>$(-10, -1, 5)$</td>
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<tr>
<td>VTR (cents)</td>
<td>120–150</td>
<td>150–171</td>
<td>300–327</td>
<td>360–400</td>
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</table>

<table>
<thead>
<tr>
<th>Common name</th>
<th>Würschmidt</th>
<th>Semisixths</th>
<th>Schismatic</th>
<th>Syntonic</th>
</tr>
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<tbody>
<tr>
<td>Comma</td>
<td>$(17, 1, -8)$</td>
<td>$(2, 9, -7)$</td>
<td>$(-15, 8, 1)$</td>
<td>$(-4, 4, -1)$</td>
</tr>
<tr>
<td>VTR (cents)</td>
<td>375–400</td>
<td>436–450</td>
<td>494–514</td>
<td>480–514</td>
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</tbody>
</table>
Two Mappings:

- Just Intonation tuning system
- Regular Temperament
- Lower Rank Tuning System
- n-dimensional Button Lattice

Temperament Mapping

Layout Mapping

characterised by
null space of comma(s)

linear and invertible
Layout Mappings

A layout mapping $\mathbf{L}$ is the physical embodiment of a mapping from a regular temperament to a button lattice.

(I) Linear layout mappings are transpositionally invariant.

(II) Conversely: transposition invariance implies linearity of the layout mapping $\mathbf{L}$ must be invertible, or else either some buttons would have no assigned note (or some notes would have no corresponding button). For fair comparisons, $\det(\mathbf{L}) = \pm 1$. 
Some Layout Matrices

Hexagonal: $B_{\text{Hex}} = \begin{pmatrix} 1.07 & 0.54 \\ 0 & 0.93 \end{pmatrix}$

Thummer: $B_{\text{Thu}} = \begin{pmatrix} 1.25 & 0.62 \\ 0 & 0.80 \end{pmatrix}$

Square: $B_{\text{Squ}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Wilson: $B_{\text{Wil}} = \begin{pmatrix} 0.94 & 0.47 \\ 0.35 & 1.23 \end{pmatrix}$

Wicki: $L_{\text{Wic}} = \begin{pmatrix} 0 & 0.54 \\ 1.86 & 0.93 \end{pmatrix}$

Wilson: $L_{\text{Wil}} = \begin{pmatrix} 5.66 & 3.30 \\ 0 & 0.18 \end{pmatrix}$

CBA-B: $L_{\text{CBA-B}} = \begin{pmatrix} 3.76 & 2.15 \\ 2.79 & 1.86 \end{pmatrix}$

Fokker: $L_{\text{Fok}} = \begin{pmatrix} 6.45 & 3.76 \\ 1.86 & 0.93 \end{pmatrix}$

CBA-C: $L_{\text{CBA-C}} = \begin{pmatrix} 3.76 & 2.15 \\ -2.79 & -1.86 \end{pmatrix}$

Bosanquet: $L_{\text{Bos}} = \begin{pmatrix} 4.90 & 2.86 \\ 0 & 0.20 \end{pmatrix}$
The Layout Matrix Defines a Swathe

\[ \mathbf{L} = (\psi \omega) = \begin{pmatrix} \psi_x & \omega_x \\ \psi_y & \omega_y \end{pmatrix} \]

As successive notes in an \( \alpha \)-reduced \( \beta \)-chain are laid onto a button-lattice they cut a *swathe* across it. The size of the swathe determines the microtonal and modulatory capabilities of the instrument; the number of \( \alpha \)-repetitions determines the overall pitch range of the instrument. The number of physical buttons on any given keyboard lattice limits the total number of intervals; the choice of layout \( \mathbf{L} \) determines the trade-off.
Swathes produced by the Wicki layout (left) and the Fokker layout (right) for $z = \frac{7}{12}$ (first row), $z = \frac{2}{3}$ (second row), $z = \frac{1}{4}$ (third row).
The Swathe

The vector position \( v_n \) of the \( n \)th note in a swathe can be expressed as a function of \( \alpha, \beta, \psi, \) and \( \omega \) as

\[
v_n = n\omega - \lfloor nz \rfloor \psi \text{ where } z = \log_{\alpha}(\beta)
\]

and \( n \in \mathbb{Z} \). The slope

\[
m = \frac{\omega_y - \psi y z}{\omega_x - \psi x z}
\]

and thickness of the swathe are given by

\[
T = \frac{1}{\sqrt{(\omega_x - \psi x z)^2 + (\omega_y - \psi y z)^2}}.
\]

Result: The higher the \( T \), the narrower it is.
Monotonic pitch axis

An *isotone* is a straight line drawn across a button-lattice that passes through the centres of buttons that produce equal pitch. The shortest distance of a button from an isotone is monotonically related to its pitch, so a line drawn at right angles to an isotone is called a *monotonic pitch axis*.

Result: An isotone has a slope equal to the swathe slope $m$.

Result: The shortest distance of a button from any given isotone is monotonically related to its pitch.
Through the Syntonic Tuning Continuum I

\[ \beta = 2^{7/4} \text{ (7-tet)} \]
Through the Syntonic Tuning Continuum II

\[ 2_{\frac{7}{12}}^4 < \beta < 2_{\frac{7}{12}} \]
\[ \beta = 2^{\frac{7}{12}} \text{ (12-tet)} \]
Through the Syntonic Tuning Continuum IV

\[ 2^{\frac{7}{12}} < \beta < 2^{\frac{10}{17}} \]
Through the Syntonic Tuning Continuum V

<table>
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<tr>
<th></th>
<th>Gb4</th>
<th>Ab4</th>
<th>Bb4</th>
<th>C5</th>
<th>D5</th>
<th>E5</th>
<th>F#5</th>
<th>G#5</th>
<th>A#5</th>
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<tr>
<td>12</td>
<td>Cb4</td>
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<td></td>
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<td></td>
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</tr>
<tr>
<td>15</td>
<td>Db4</td>
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<td></td>
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</tr>
<tr>
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<td>Ab3</td>
<td>Bb3</td>
<td>C4</td>
<td>D4</td>
<td>E4</td>
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<td>17</td>
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</tbody>
</table>

\[ \beta = 2^{10} (17\text{-tet}) \]
Through the Diatonic Tuning Continuum VI

\[ 2^{\frac{17}{10}} < \beta < 2^{\frac{3}{5}} \]
Through the Syntonic Tuning Continuum VII

\[ \beta = 2^\frac{3}{5} \text{ (5-tet)} \]
...work in progress...

But won’t these weird temperings sound horribly dissonant?

In the same way that JI is related to the harmonic spectra (through the process of generating a dissonance curve with minima that lie at the desired scale steps), so the tempered intonations can be related to spectra with tempered partials.

The overtones of a sound can be matched to the temperaments in a straightforward way using the generators. The dissonance curves of these tempered harmonics have minima at the locations of the primary consonances of the related temperaments.

We can change the spectrum of the sounds along with the tunings!
Possible mappings (using 5-limit syntonic JI) for the harmonics as a function of the generators $\alpha$ and $\beta$:

- $O = \alpha$ (octave)
- $F = \beta\alpha^{-1}$ (fifth)
- $T = \alpha^{-6}\beta^4$ (third)

<table>
<thead>
<tr>
<th></th>
<th>2, 3, 5</th>
<th>jO, jF, jT</th>
<th>O, F, T</th>
<th>O, F, S</th>
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<tr>
<td>1</td>
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<td>1</td>
<td>1</td>
<td>1</td>
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<td>2</td>
<td>jO</td>
<td>O</td>
<td>O</td>
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<td>3</td>
<td>jO jF</td>
<td>O F</td>
<td>O F</td>
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<td>4</td>
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<td>jO^2</td>
<td>O^2</td>
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<td>jO jT</td>
<td>O T</td>
<td>O F S</td>
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<tr>
<td>6</td>
<td>2^3 3^1</td>
<td>jO^2 jF^1</td>
<td>O^2 F</td>
<td>O^2 F</td>
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<td>7</td>
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<td>O^3</td>
<td>O^3</td>
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<td>O^2 F^2</td>
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<tr>
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<td>O^2 T</td>
<td>O^2 F S</td>
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<tr>
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<td>1^1</td>
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<td>12</td>
<td>2^2 3^1</td>
<td>jO^3 jF^1</td>
<td>O^3 F</td>
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<tr>
<td>14</td>
<td>2^1 7^1</td>
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</tr>
<tr>
<td>15</td>
<td>3^1 5^1</td>
<td>jO^3 jF^1 jT</td>
<td>O^3 F^1 T</td>
<td>O^2 F^2 S</td>
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<tr>
<td>16</td>
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<td>jO^4</td>
<td>O^4</td>
<td>O^4</td>
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</table>
Dissonance curves for the various spectra created from the generators over the syntonic continuum have minima at the required primary consonances.
A I-IV-V-I progression annotated in several tunings throughout the syntonic continuum. Listen to the progressions with the tempered spectra.
Prototype “Thummer” Keyboard
Summary

It is possible to design keyboards capable of smoothly moving among a continuum of tunings, retaining the same fingerings in all keys over the continuum. This uses a parameterization based on commas.

The Valid Tuning Range can be easily calculated in terms of a set of privileged intervals (e.g., the primary consonances)

Linear Layout maps can be understood in terms of properties of the swathe (slope and thickness) and of a monotonic pitch axis.

It is also possible to modify the spectra of sounds so that a degree of consonance can be maintained throughout the continuum.