

Inharmonic strings and the hyperpiano

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Abstract

This paper describes a design procedure for a musical instrument based on inharmonic (nonuniform) strings. Fabricating nonuniform strings from commercially available strings constrains the possible string diameters, and hence the possible inharmonicities. Detailed simulations of the strings are combined with a measure of sensory dissonance (or roughness) to help narrow down the remaining possibilities. A particularly intriguing variation is a string that consists of three segments: two equal unwound segments surrounding a thicker wound portion. The corresponding musical scale, built on the 12th root of 4, is called the *hyperoctave*. A standard piano is modified to play in this tuning using these inharmonic strings; this instrument is called the *hyperpiano*.

Keywords: vibrations of strings, nonideal strings, musical instrument design, inharmonic oscillations

1 Ideal and non-ideal strings

An ideal string has uniform mass density, negligible stiffness, and is placed under enough tension that the effects of gravity may be ignored. In this ideal setting, the string vibrates in a periodic fashion and the overtones of the spectrum are located at exact multiples of the fundamental period, in accordance with the one-dimensional linear wave equation [1]. The inevitable small deviations from these idealized assumptions cause a small amount of inharmonicity which is often cited as the reason that a piano is tuned with stretched octaves, that is, where the octave is tuned to a factor slightly smaller than two in the low frequency regions and slightly larger than two in the high frequency regions [2]. Large deviations from these assumptions can be achieved using

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the techniques of the “prepared piano” where weights (and other objects) are placed on (or in contact with) the string, effectively causing the vibrating element to have nonuniform density. The sound of such preparations has been described as bell-like, metallic, knocking, buzzing, thudding, and gong-like; the sounds are characterized by overtones that are not related in a simple harmonic fashion and are often idiosyncratic to the particular installation and preparation [3].

In between these “large” and the “small” deviations from the ideal lies a middle ground that is relatively unexplored, where the nonuniformity of the density of the strings is carefully studied and controlled in order to achieve a desired musical effect. This paper explores one possible musical system (the *hyperoctave*) based on an inharmonic string, and provides tools that may aid in the discovery and study of other such systems. The invention of new musical instruments and ways to tune and play them has a long history [4, 5] and continues to the present, though modern approaches are often based on digital rather than analog sound production [6]. From an acoustical point of view, the idea of designing an instrument based on an inharmonic sound contrasts with the more common approach of beginning with an inharmonic vibrating element and trying to make it more harmonic: the ongoing work to try and manufacture harmonic bells [7, 8, 9] is perhaps the best known example. But even exotic instruments such as the bent metal bars of the *alembe* [10] can be coerced into approximating a harmonic series with the right sizes and angles, and it is now possible to tune the overtones of a percussion ensemble [11].

While there is no conceptual difficulty in imagining a string with an arbitrary density profile, it is not easy to fabricate strings with oddly varying contours. Perhaps the simplest situation is shown in Fig. 1(a) where a nonuniform string is conceptualized as consisting of a sequence of connected segments, each of which is uniform. Though it is straightforward to specify such a sequence and to mathematically calculate the way it would sound (as in Section 2), there is no easy way to manufacture such a string for arbitrary mass densities ν_1, ν_2, \dots .

But for certain specific densities, and for a small number of segments (either two or three), it is possible to exploit the structure of commonly manufactured strings. A wound string, as shown schematically in Fig. 1(b), consists of a core metal wire with mass density μ_2 surrounded by a second wire wrapped around the outside (shown with a density of μ_1). Stripping away the winding from a portion of the string effectively creates a two-segment string that is readily available from commercial sources. Stripping away the winding from both ends of the string effectively creates a three-segment string (though with the constraint that the first and third segments have the same density). The strategy followed in this paper is to look at the parameters μ_1 and μ_2 that are readily available, and to pick those that may lead to useful musical structures.

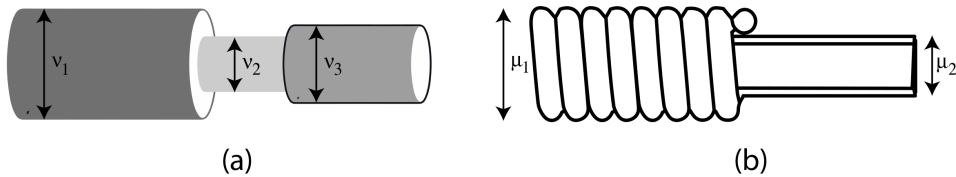


Figure 1: A nonuniform string can be thought of as a sequence of connected uniform strings, shown schematically in (a) with three segments ν_1 , ν_2 and ν_3 . A typical round-wound string (b) commonly used for guitar and piano strings has a solid metal core with width μ_2 inside a winding with width μ_1 .

Given an inharmonic string, one way to characterize the quality of the resulting musical system is to draw the dissonance curve [12] as described in Section 3. This provides a way of locating the intervals that are maximally consonant, where consonance is taken to be the inverse of sensory dissonance or roughness. Indeed, several examples [13] of such systems are shown, but most require sophisticated electronic sound synthesis. The use of inharmonic strings provides an acoustic analog of this digital sound synthesis. Section 4 shows how a set of closely related nonuniform strings, of the easily manufactured form in Fig. 1(b), can be specified and matched to each other and to a specific tuning and then matched in scale [14]. By making judicious choices, the resulting system can operate formally in an analogous way that the ideal harmonic string corresponds to a just intonation (or to its 12-tone equal-tempered approximation). Section 4 presents the hyperoctave system, which is a tuning that has its unit of repetition at the double octave (making it analogous to the Bohlen-Pierce scale [15, 16], which has its unit of repetition at the interval 3:1). The construction of the hyperpiano, an acoustic instrument that embodies this inharmonic structure and unusual scale, is then described in detail in Section 5, and some sound examples are presented.

2 Spectrum of an inharmonic string

This section follows the development in Kalotas and Lee [17], which provides a straightforward way to determine the spectrum of the sound of a segmented string such as in Fig. 1(a). The physical properties of each segment j are characterized by a length ℓ_j and a wave number

$$k_j = \omega \sqrt{\frac{\mu_j}{T}}, \quad (1)$$

where T is the tension of the string, ω is frequency, and μ_j is the mass density of the segment. The wave equation within a segment can be represented using the matrices

$$Z(k, \ell) = \begin{pmatrix} \cos(k\ell) & -\sin(k\ell)/k \\ k \sin(k\ell) & \cos(k\ell) \end{pmatrix} \quad (2)$$

and the product of n matrix segments is

$$F(\omega) = \prod_{j=1}^n Z(k_j, \ell_j). \quad (3)$$

Guaranteeing continuity across the segments (of both the displacements and their derivatives) and taking into account the end conditions (that the string displacement at the two ends must always be zero) leads to the condition that the frequencies of the modes of vibration occur where the $\{1, 2\}$ element of the matrix $F(\omega)$ is zero, that is, for those ω with

$$F_{12}(\omega) = 0. \quad (4)$$

For example, with a single segment of length $\ell_1 = L$ and with wave number $k_1 = \omega\sqrt{\frac{\mu_1}{T}}$,

$$F_{12}(\omega) = Z_{12}(k_1, \ell_1) = \frac{-\sin(k_1\ell_1)}{k_1} = \frac{-\sin(\omega\sqrt{\frac{\mu_1}{T}}L)}{\omega\sqrt{\frac{\mu_1}{T}}} = 0$$

at all integer multiples of $\omega = \frac{\pi}{L}\sqrt{\frac{T}{\mu_1}}$, that is, at all frequencies of a harmonic series. With two segments that have lengths $\ell_1 + \ell_2 = L$ and wave numbers k_1 and k_2 , the roots of

$$\begin{aligned} F_{12}(\omega) &= \{Z(k_1, \ell_1)Z(k_2, \ell_2)\}_{12} \\ &= -\frac{\sin(k_1\ell_1) \cos(k_2\ell_2)}{k_1} - \frac{\cos(k_1\ell_1) \sin(k_2\ell_2)}{k_2} \end{aligned} \quad (5)$$

cannot be calculated in closed form, but they can be readily solved numerically. For example, the built in numerical solver in Mathematica has no trouble returning correct answers for 12-15 segments. The generalization to three segments is straightforward, though the equations become more cumbersome as the number of segments increases.

Experiment 1 *A nonuniform string mounted on a monochord has length 0.9144 m. The heavier wound half ($\ell_1 = 0.457$ m) has a mass of 12.6 grams ($\mu_1 = 0.0275591$ kg/m), and the lighter unwrapped half ($\ell_2 = 0.457$ m) weighs 3.3 grams ($\mu_2 = 0.00721785$ kg/m). Fig. 2(a) shows the predictions of the model by plotting $F_{12}(\omega)$ of (5). The zeros of this function are listed in the figure caption. The string was plucked, the sound was recorded, and an FFT was calculated to measure the spectrum of the*

resulting sound, which is shown in Fig. 2(b). There is good agreement between the theoretical locations of the overtones (as given by the roots of (5)) and the frequencies of the spectral peaks. Details of these calculations can be found in the Mathematica notebook `monochord.nb`. All computer code, sound files, and supporting material can be found at the paper website [18].

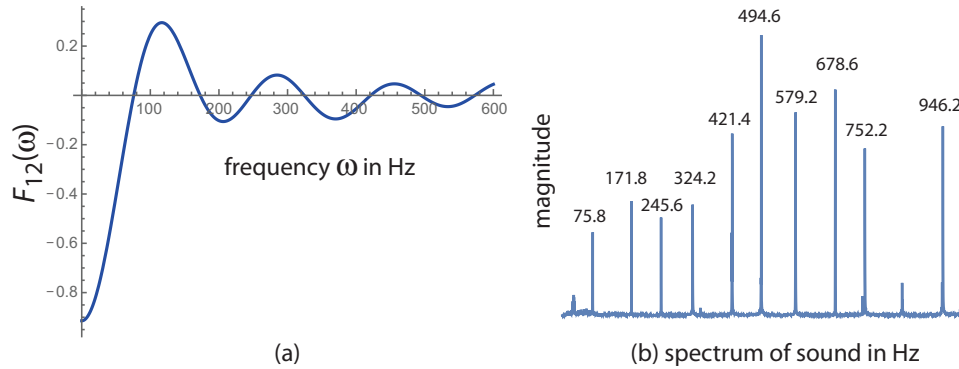


Figure 2: The function $F_{12}(\omega)$ of (5) is shown in (a). The first eight zeros of this function occur at 75.7979, 172.73, 247.146, 324.379, 421.199, 494.303, and 573.009 Hz. The magnitude spectrum of the sound produced by the string is then calculated using an FFT. The agreement between the theoretical model and the experiment is striking.

Experiment 1 gives a measure of confidence in the use of the approximation that the part-wound part-unwound string of Fig. 1(b) can be successfully approximated by two uniform segments as in Fig. 1(a). We conducted a variety of similar checks throughout the design procedure.

Experiment 2 Using the same string densities μ_1 and μ_2 as in Experiment 1, this experiment simulates all the possible combinations of wound vs. unwound lengths. Suppose the total length of the string is L . Let $0 \leq \lambda \leq 1$ specify the lengths of the two segments so that $\ell_1 = (1-\lambda)L$ and $\ell_2 = \lambda L$. When $\lambda = 0$ the string is all of density μ_1 , when $\lambda = 1$ the string is all of density μ_2 , and $\lambda = 1/2$ corresponds to the conditions of Experiment 1. Fig. 3 shows the interface of the notebook `monochordSim.nb`, which allows interactive specification of λ . The “play” button creates a simple additive synthesis simulation of the sound represented by the inharmonic string at the selected value of λ . The set of numbers on the right shows all the zeros of $F_{12}(\omega)$ up to 2K Hz.

When the λ -slider in Experiment 2 is all the way to the left, the simulation represents a uniform light gauge string. When λ is at the right, it represents a uniform

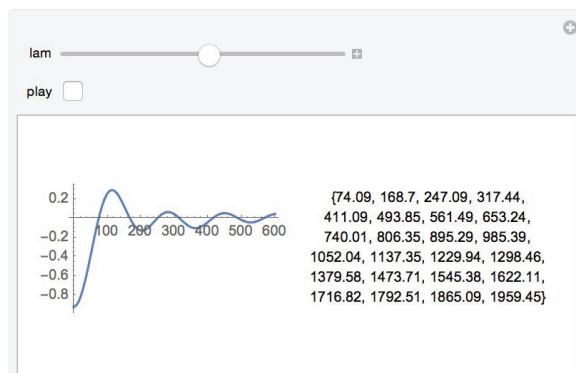


Figure 3: The inharmonic string with μ_1 and μ_2 from Experiment 1 can be adjusted to any ratio and the sound synthesized (via a simple additive synthesis routine). The pitch descends as the slider moves from left to right.

heavier gauge string. This is why the fundamental (the lowest frequency) is clearly discernible at the two extremes (because the sound is harmonic). As a consequence, the pitch descends smoothly over the almost-octave from 122 Hz down to 62.5 Hz. This is a ratio of $\frac{122.14}{62.5} = 1.95424$. This ratio arises because of the specific μ values used since $\sqrt{\frac{\mu_1}{\mu_2}} = 1.95402$. Experiment 2 may lead to the (false) impression that it is easy to predict the pitch of the inharmonic sounds. To test this concretely, we conducted

Experiment 3 *With the same inharmonic string parameters as in Experiments 1 and 2, and the same interface as in Experiment 2, this experiment changes the tension parameter T . For each value of λ , the tension is chosen so that the lowest frequency will always be 100 Hz. A Mathematica notebook that implements this is `monochordSimPitch.nb`. If the pitch is determined by the lowest fundamental, the pitch should remain fixed for all values of λ .*

Even a cursory perusal of the sounds in Experiment 3 shows that the pitch seems to go up and down somewhat arbitrarily as λ changes even though the lowest partial is always locked at 100 Hz. This means that the pitch is *not* determined by the fundamental, but by some combination of the upper partials.

3 Calculating sensory dissonance

The psychoacoustic work of R. Plomp and W. J. M. Levelt [19] provides a basis on which to build a measure of sensory dissonance that can be used to guide the design of the strings. In their experiments, Plomp and Levelt asked volunteers to rate the

perceived dissonance or roughness of pairs of pure sine waves. In general, the dissonance is minimum at unity, increases rapidly to its maximum somewhere near one quarter of the critical bandwidth, and then decreases steadily back towards zero. When considering timbres that are more complex than pure sine waves, dissonance can be calculated by summing up all the dissonances of all the partials, and weighting them according to their relative amplitudes. For harmonic timbres, this leads (as expected) to curves having local minima (or points of local maximum consonance) at small integer ratios (as in Fig. 4), which occur near many of the steps of the 12-tone equal tempered scale. Similar curves can be drawn for inharmonic timbres [12], though the points of local consonance are generally unrelated to the steps and intervals of the 12-tone equal tempered scale.

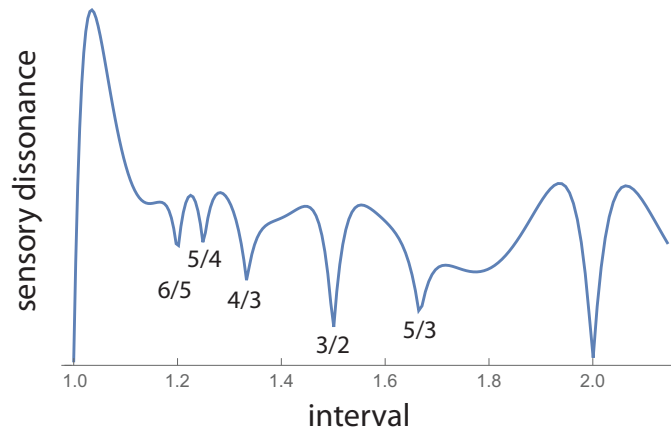


Figure 4: A dissonance curve is a plot of the function $D_{FF}(\alpha)$ of (8). This figure assumes a harmonic sounds with six equal partials. As shown in [19], such a curve has minima at many of the simple integer ratios.

To be concrete, the dissonance between a sinusoid of frequency f_1 with amplitude v_1 and a sinusoid of frequency f_2 with amplitude v_2 can be parameterized as

$$d(f_1, f_2, v_1, v_2) = \max(v_1, v_2) \left(e^{-as|f_2-f_1|} - e^{-bs|f_2-f_1|} \right) \quad (6)$$

where

$$s = \frac{d^*}{s_1 \min(f_1, f_2) + s_2}, \quad (7)$$

$a = 3.5$, $b = 5.75$, $d^* = .24$, $s_1 = .021$ and $s_2 = 19$ are determined by a least squares fit to the averaged perceptual data gathered from the listening tests. The amplitude terms are usually taken to be in dB and the functional form $\max(v_1, v_2)$ ensures that

softer components contribute less to the total dissonance measure than those with larger amplitudes. The parameters s and d^* define the interval where maximum dissonance occurs, which is a function of the critical bandwidth. Thus (7) can be interpreted as allowing a single functional form to smoothly interpolate between a family of dissonance curves, each of which is valid in a particular frequency range [12].

More generally, a (complex) timbre F is a collection of n sine waves (or partials) with frequencies (f_1, f_2, \dots, f_n) and amplitudes (v_1, v_2, \dots, v_n) , where the f_i are typically ordered from lowest to highest. Similarly, let G be a collection of m sine wave partials with frequencies (g_1, g_2, \dots, g_m) and amplitudes $(\nu_1, \nu_2, \dots, \nu_m)$. The sensory dissonance between F and G at an interval α is the sum of the dissonances of all pairs of partials

$$D_{FG}(\alpha) = \sum_{i=1}^n \sum_{j=1}^m d(f_i, \alpha g_j, v_i, \nu_j). \quad (8)$$

There are many ways to look at dissonance curves of the inharmonic strings from Section 2. The next three experiments show several possibilities: Experiment 4 varies the relative lengths of the two segments, Experiment 5 varies the densities of the two segments, and Experiment 6 generalizes to 3-segment strings.

Experiment 4 *Using the same string densities μ_1 and μ_2 as in Experiment 1, this experiment draws the dissonance curve between two strings, each of which has a parameter to specify the relative lengths of the two segments (as in Experiment 2). Thus the two length parameters for the first string $\ell_1^1 = (1 - \lambda_1)L$ and $\ell_2^1 = \lambda_1 L$, and the length parameters for the second string $\ell_1^2 = (1 - \lambda_2)L$ and $\ell_2^2 = \lambda_2 L$, may be adjusted independently. Fig. 5(a) shows the interface of the notebook `lambdaTwoStrings.nb`, which allows interactive specification of the two λ parameters and displays the resulting dissonance curve over a little more than the two octaves, i.e., over the range $0.4 < \alpha < 2.2$. The “link” button forces the two λ values to be equal, and the “number of spectral lines” slider changes the number of overtones each string is assumed to contain.*

Experiment 5 *This experiment varies the values of μ_1 and μ_2 , and draws the resulting dissonance curves. Both strings have the same λ , and the resulting curves show how the dissonance curves change as a function of the densities. Fig. 5(b) shows the interface of the notebook `varyingDensities.nb` and one possible set of parameter values.*

Experiment 6 *This experiment considers inharmonic strings formed in three segments: an unwound (thin) segment, a wound (thick) segment, and an unwound (thin) segment. The resulting dissonance curves can be drawn as in Fig. 6, which shows the interface from the notebook `threePartString.nb`.*

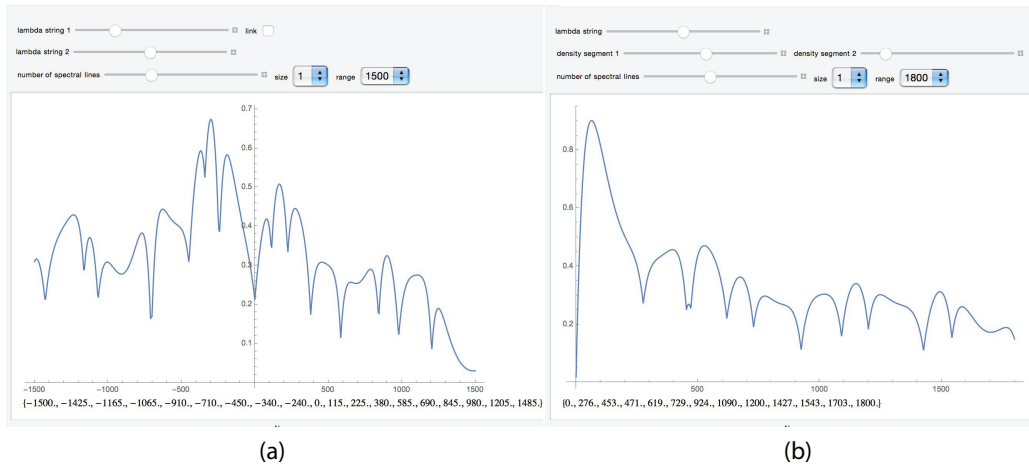


Figure 5: (a) The dissonance curve for two inharmonic strings (each with μ_1 and μ_2 as in Experiment 1) can be adjusted to any relative lengths using the λ sliders. (b) The dissonance curve for two inharmonic strings both with the same λ but with varying densities μ_1 and μ_2 . In both cases, the curves update as the sliders move, and the locations of the minima (within the specified range) are shown in cents at the bottom.

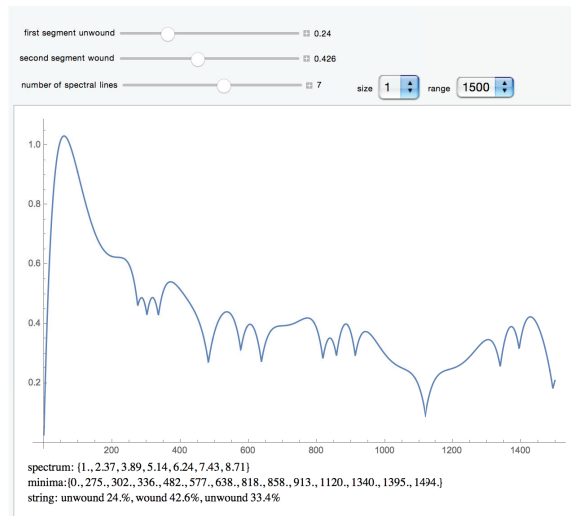


Figure 6: The dissonance curve for an inharmonic string formed from three segments with mass densities μ_1 , μ_2 , and μ_1 . The dissonance curve updates as the sliders move, and the peaks of the spectrum (normalized so that the fundamental is frequency 1), the locations of the minima (in cents), and the string proportions (the values of ℓ_1 , ℓ_2 , and ℓ_3 , expressed as percentages) are shown at the bottom.

4 The hyperoctave: a musical system based on inharmonic strings

The laws of physics impose certain limitations on the spectrum of a nonuniform string. Equally important (though less fundamental) are the limitations associated with the current state of the art in string manufacturing. In principle it is possible to fabricate almost any specified nonuniform contour, but considerations of cost and practicality dictate the use of standard string widths and windings. Given these constraints, the goal is to design a musical instrument together with a way of playing that instrument (i.e., a “musical system”) that can highlight the unique sound quality of the inharmonic string and emphasize the privileged intervals that occur at the minima of the dissonance curve.

As Experiments 4–6 suggest, there are many choices. We looked at, listened to, and studied many different possibilities (using the interactive notebooks, simulations, and computer programs), and eventually selected one for detailed investigation. The chosen system has a dissonance curve that mimics the shape of the curve of the harmonic string (as in Fig. 4) when stretched out over two octaves (see Fig. 7). Thus we call it the *hyperoctave* system, meaning over or above the octave. The piano-like instrument constructed in Section 5 is called the *hyperpiano*, and we use the same prefix whenever convenient. For example, when applied to intervals, hyper indicates a doubling of frequency. Thus a hyperoctave is 2400 cents, a minor hypersecond is 200 cents, and a hyperfifth is 1400 cents, etc.

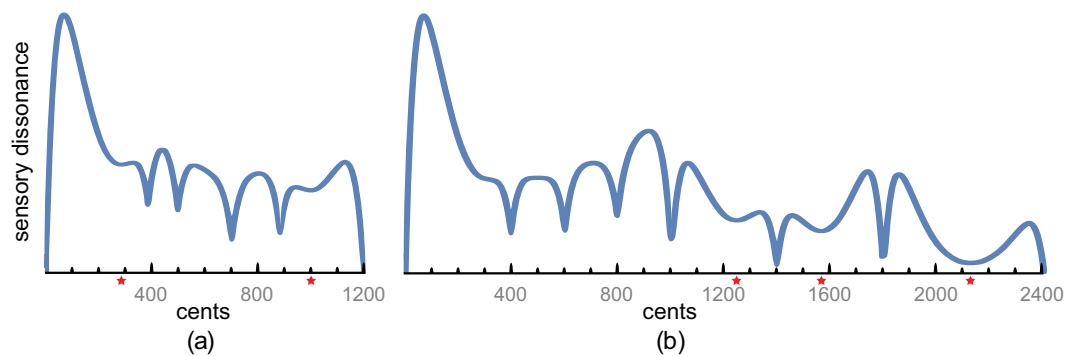


Figure 7: The dissonance curve for the first five partials of a uniform string (a) and the dissonance curve for the first five partials of a hyperoctave nonuniform string (b). The broad minima at the starred locations are not caused by coinciding partials.

There are a number of similarities between the hyperoctave system and the harmonic system, as shown in Fig. 7. For example, focusing on the minima formed by

coinciding partials, the four deepest minima (other than the unison) of the harmonic dissonance curve are the octave, perfect fifth, 5-limit major sixth, and perfect fourth. This is directly analogous to the hyperoctave system which has its deepest minima at the hyperoctave, perfect hyperfifth, major hypersixth, and perfect hyperfourth. The pattern diverges at this point because the fifth deepest minimum for the uniform string is the 5-limit major third, while the fifth deepest minimum for the nonuniform string is the major hypersecond (which are, coincidentally, nearly the same interval). It is also interesting to note that the minima for the nonuniform string fall closer to the traditional 12-tone equal-tempered scale steps than the minima for the uniform string. Together, these show that the hyperoctave system has a large degree of contrast between its consonant and dissonant intervals, which may aid in the perception of movement between tension and relaxation in tonal music, and offers hope that its odd form of tonality can be a fruitful source of musical inspiration. Table 1 summarizes many of the intervallic properties of the non-broad minima of the dissonance curves for the two systems.

Uniform String		
ratio	cents	name
5/4	386	5-limit major third
4/3	498	perfect fourth
3/2	702	perfect fifth
5/3	884	5-limit major sixth
2/1	1200	octave
Nonuniform String		
5.06/4.02	398	major hypersecond
4.02/2.84	602	minor hyperthird
2.84/1.79	799	major hyperthird
5.06/2.84	1000	perfect hyperfourth
4.02/1.79	1401	perfect hyperfifth
5.06/1.79	1799	major hypersixth
4.02/1.00	2409	hyperoctave

Table 1: The non-broad minima of the dissonance curves in Fig. 7 are shown for the uniform string with spectrum $\{f, 2f, 3f, 4f, 5f\}$ and for the nonuniform string with spectrum $\{f, 1.79f, 2.84f, 4.02f, 5.06f\}$.

The fundamental unit of repetition in the harmonic series and in the resulting dissonance curve of Fig. 7(a) is the octave, while the fundamental unit of repetition in Fig. 7(b) is the hyperoctave. This suggests that just as 12-tone equal temperament (12-

tet, based on steps of equal size $\sqrt[12]{2}$) forms a basis on which to build octave-repeating scales such as common practice major and minor scales, 12-hypertone equal temperament (12-het, based on steps of size $\sqrt[12]{4}$) forms a basis on which to build hyperoctave-repeating scales. These are compared in Fig. 8. Interestingly, the octave is notably dissonant for the nonuniform hyperoctave string. This is arguably beneficial, since it forms a hypertritone within the hyperoctave system, and helps to distinguish the two systems.

The hyperoctave system can be viewed as an inharmonic analog of the Bohlen-Pierce scale [15] which is designed to be played on instruments with a spectrum consisting of odd harmonics, and which divides the octave+fifth into 13 equal parts (in steps of size $\sqrt[13]{3}$). Dissonance curves for the Bohlen-Pierce spectrum drawn in [13] show minima that occur at many of the $\sqrt[13]{3}$ scale steps. Bohlen has also investigated a scale based on steps of size $\sqrt[12]{3}$ that is inspired by the 4:7:10 chord (called A12 by Moreno [20]). This showcases a systematic developmental progression from $\sqrt[12]{2}$, to $\sqrt[12]{3}$, to $\sqrt[12]{4}$. Pseudo-octave scales (such as the 3-based systems) do not typically benefit from the perception of octave equivalency; the repetition of the hyperoctave scale at the double-octave may aid in the perception of hierarchical relations within its tonal system.

Observe that 12-het can be viewed as a concatenated pair of traditional whole-tone scales. This raises the possibility that the hyperoctave system may function like the whole-tone scale. Figure 8 shows two octaves of the dissonance curve for the uniform string in the top plot and the dissonance curve for the nonuniform string in the bottom plot. The light vertical lines and dashes show how points of maximum consonance of one system tend to fall near points of maximum dissonance of the other. This suggests that even though many intervals may be similar between the two systems, they will function quite differently. This supports the potential for cadences within the hyperoctave system which would be impossible within the harmonic system, and vice versa.

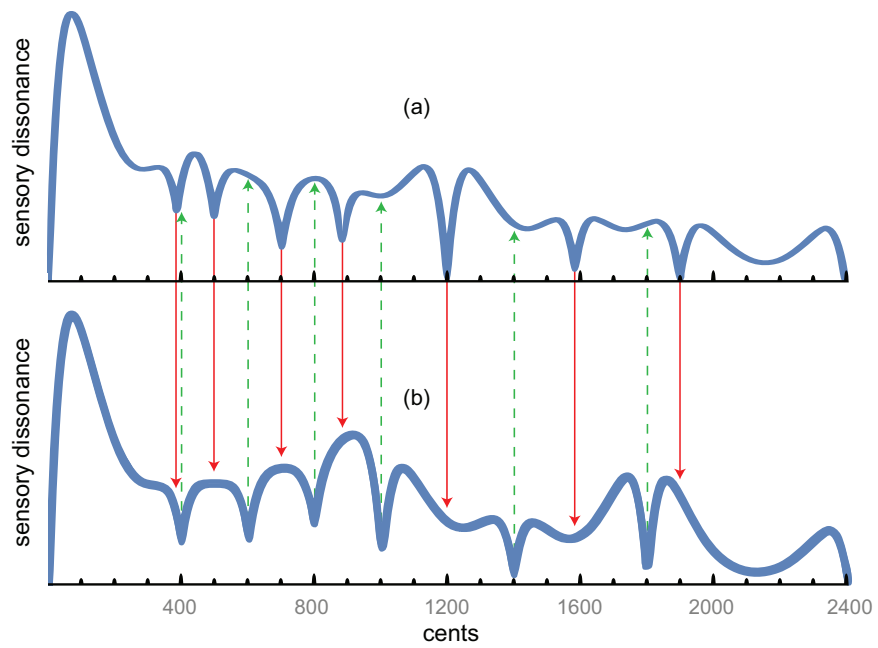


Figure 8: (a) shows the dissonance curve for a harmonic string with five partials drawn over a two octave range. (b) shows the dissonance curve for the hyperoctave nonuniform string with five partials, plotted over the same range. The vertical lines emphasize how the minima of the top curve tend to occur near maxima of the bottom (and vice versa), suggesting that the tonal functions of the two systems may be very different.

5 The hyperpiano

To translate theory into practice, several choices must be made involving musical notation, keyboard specification and design, and pitch reference. The dissonance curve in Fig. 7(b) suggests that the most important intervals in the hyperoctave system are located at the values given in Table 1. These can be generalized to a complete tuning for the instrument by having keys play all pitches of the $\sqrt[12]{4}$ chromatic scale. Let the prefix v represent the parts of the hyper system (chosen because v is the first letter in the Greek root of the English word hyper) and designate Middle C (261.626 Hz) as the enharmonic equivalent of Middle v C. The top keyboard in Fig. 9 shows how these pitches would occur on a standard piano.

A used Emerson grand piano was purchased, and all of the strings were removed. Specific keys were also removed from the keyframe in accordance with the adapted keyboard depiction in Fig. 9 (a stretched arrangement of this nature is necessary, since the cast-iron frame was originally scaled for 12-tet). The speaking length of each applicable string was measured by running hemp cord between the agraffes and the capodastro bar (near the pin block), to the tuning pins (near the hitch pins). This data was then used to determine suitable steel piano wire gauges (for the treble register) and copper-wound steel piano wire gauges (for the bass register).

The tension T (in pounds¹) on a standard piano string can be calculated [21, 22] as

$$T = 2^{m/6} \left(\frac{Ld}{802.6} \right)^2 (1 + B) \quad (9)$$

where m is the piano key number (as arranged on the traditional keyboard in Fig. 9), L is the speaking length of the string in inches, and d is the piano wire diameter in mils. The wrap weighting factor

$$B = 0.89 \left(\frac{D^2}{d^2} - 1 \right) \quad (10)$$

can be calculated from D , which is the overall diameter of the copper-wound piano wire in mils (for unwound piano wire $B = 0$). The maximum safe string tension is $T_{max} = 0.557d^{1.667}$. Although these two equations pertain specifically to uniform strings, the hyperoctave string is uniform along 90.4% of its length, and the remaining 9.6% is 3.818 times as dense; consequently, it seems safe to assume that (9) and (10) provide reasonable approximations.

Beginning with the treble register (traditional key numbers 38–64, adapted key numbers 37–64), only two piano wire diameters are needed: 0.033" and 0.037". Together, these provide an average string tension of 139 pounds. The ideal hyperoctave

¹The literature on piano strings and stringing, as well as the manufacturer's specifications are written primarily in English units. We will translate this to metric units if it is the policy of the journal.

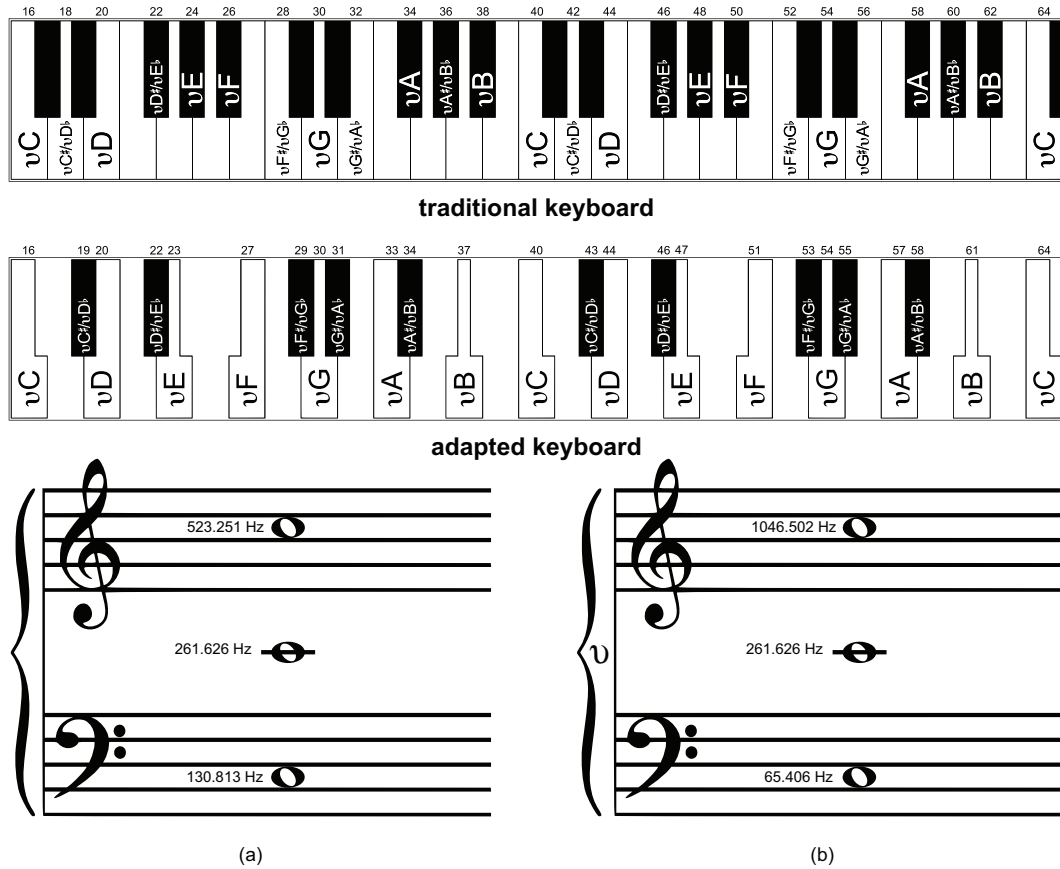


Figure 9: (a) depicts a traditional grand staff with three frequencies given: Middle C (261.626 Hz), one octave below Middle C (130.813 Hz), and one octave above Middle C (523.251 Hz). (b) shows a grand hyperstaff with three frequencies: Middle v C (261.626 Hz), one hyperoctave below Middle v C (65.406 Hz), and one hyperoctave above Middle v C (1046.502 Hz). Above the two staves are two keyboard drawings. The upper drawing is a traditional seven-plus-five keyboard with a hyperchromatic scale displayed across two hyperoctaves. The lower keyboard is a modification with a hyperchromatic scale displayed across two hyperoctaves. Piano key numbers for each hyperchromatic scale are provided above each keyboard.

string requires the first 12% of the speaking length to reside at a given uniform density. The next 9.6% of the speaking length must then reside at a uniform density 3.818 times that of the first 12%. And the remaining 78.4% of the speaking length must match the density of the original 12%. In order to achieve this density distribution with the 0.033" and 0.037" piano wires, in accordance with the current state of established string manufacturing processes, they must each be wound with the correct gauge of copper wire (on a string-winding machine), and then portions of copper must be manually unwound by hand so that only 9.6% of the speaking length remains wound. Piano wire is often manufactured with a circular cross-section, but a hexagonal cross-section is also available. When copper is wound around hexagonal piano wire, the edges formed by the six vertices tend to bite into the inner side of the copper wrap wire, thus locking it into place. This is advantageous for our purposes, as it helps to prevent the wrap wire from uncoiling during the unwinding process.

Two 6" hexagonal wires were modeled in the AutoCAD software [23]: one for each of the diameters (the "diameter" of hexagonal wire is specified as though a hexagonal cross-section would circumscribe about a circle of the desired diameter). The `massprop` command was then used to estimate the volume of each solid. The estimated volume of the 0.033" diameter wire was 0.09273 cm³, and the estimated volume of the 0.037" diameter wire was 0.11657 cm³. Piano wire is generally ascribed a density of 7.85 g/cm³. Taken together, these figures predict a mass of 0.728 g for the 0.033" wire, and a mass of 0.915 g for the 0.037" wire. The desired copper wrap density for each hexagonal wire prediction can be calculated from

$$W = CR - C \tag{11}$$

where C is the mass of a given length of hexagonal core wire, W is the mass of an equal length of wound copper wire, and R is the density ratio (i.e., 3.818). Thus the copper wire wound around 6" of 0.033" hexagonal wire has a mass of 2.052 g and the copper wire wound around 6" of 0.037" hexagonal wire has mass 2.578 g.

To calculate the diameter of the ideal copper wrap wires, we turn again to AutoCAD. Helices with circular cross-sections (of random diameter) were extruded around the hexagonal wires, so as to model wrap wire. In order to simulate a tight wrap (one in which the hexagonal wires bite into the copper wires), the wrap wire was wound as though it were circling a cylinder extruded from a circle inscribed within a cross-section of each hexagonal wire. The volume of each helix was estimated (using the `massprop` command), the estimates were compared against the density of annealed copper (8.9 g/cm³), and the resulting masses were calculated. The helices were then deleted and replaced with new helices in which the cross-section diameters were adjusted in an attempt to more closely match the desired wrap wire mass. This trial-

and-error approach continued until the desired density relations were predicted. The predictions show that if we allow the wire diameters to be taken to the ten-thousandths place, then a 6'' length of 0.033'' hexagonal wire wound with 52.869'' of 0.0184'' copper wire (326.087 turns) will possess a mass of 2.05 g – only 2 mg less than the desired mass. Likewise, a 6'' length of 0.037'' hexagonal wire wound with 52.918'' of 0.0206'' copper wire (291.262 turns) will possess a mass of 2.572 g – only 6 mg less than the desired mass. The predicted ideal wire diameters are depicted in Fig. 10.

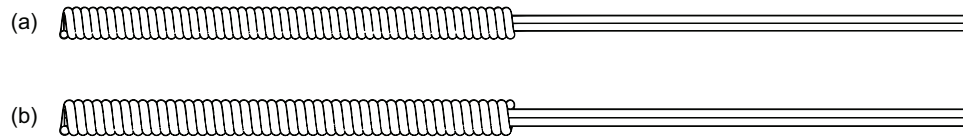


Figure 10: (a) Half of a 2'' length of 0.033'' hexagonal piano wire is wound 54.348 times with a 0.0184'' copper wire. The predicted density ratio for the two segments of this string is 3.816:1. (b) Half of a 2'' length of 0.037'' hexagonal piano wire is wound 48.544 times with a 0.0206'' copper wire. The predicted density ratio for the two segments of this string is 3.811:1. These represent a predicted best-case scenario for wire diameters (accurate to the ten-thousandths place).

The Mapes Piano String Company [24] was contacted. In their inventory, the best match to 0.0184'' copper wire is 0.0181'' and the nearest match to 0.0206'' copper wire is 0.0204''. Five 0.033'' hexagonal core strings were commissioned, each to be wound with one of the five nearest 0.0184'' copper wire matches, and five 0.037'' hexagonal core strings were also commissioned, each to be wound with one of the five nearest 0.0206'' copper wire matches. Once the strings arrived, they were cut, weighed, and their density ratios were calculated. The best copper wrap wire candidate for the 0.033'' hexagonal piano wire was found to be 0.0181'' (yielding a density ratio of 3.675:1), and the best copper wrap wire candidate for the 0.037'' hexagonal piano wire was found to be 0.0204'' (yielding a density ratio of 3.735:1). The above procedures were then repeated to determine the best parameters for the bass register strings. One difference is that the hexagonal cores in the bass register needed to be double wound. Therefore, when applying (11), the first winding and the core form an analog to C while the second winding forms the analog to W .

The Emerson piano has a modified keyboard as in Fig. 9 and strings as discussed above. The final hyperpiano is shown in Fig. 11 and can be seen “in action” in Sound Example 1 which can be found online at [18].



Figure 11: The completed hyperpiano (color online)

6 Sound examples

Sound Example 2 (all examples are available online at [18]) is a high-fidelity recording of Sound Example 1. Sound Examples 2 and 3(c) were recorded with two Shure SM57s (in stereo) and an EV RE320 (set directly above the soundboard). The microphones were routed into a Focusrite Scarlett 18i20 and then into Logic (audio software). Digital equalization was used to remove frequencies below 20 Hz and above 20 kHz, and Logic’s Stereo Spread plugin was used to avoid phase issues. The Slate Digital FG-X mastering plugin was used for light compression, leveling, and overall gain monitoring. It consists of a brief succession of melodic runs demonstrating the complete tessitura of the hyperpiano. The sustain pedal is depressed throughout the entire example in order to allow the spectrum of each note to resonate freely with the spectra of every other interrelated note in the system (consequently disclosing the overall tone quality of the system). The example begins with Middle ν C and ascends through the white keys above Middle ν C, after which it reverts back to Middle ν C and descends through the white keys below Middle ν C. It then ascends through the black keys, but only to descend once more – this time through every single key.

Sound Examples 3(a), (b), and (c) consist of a series of cadential progressions as shown in Fig. 12. Audio samples of a Steinway Model D grand piano, derived from [25], were used to implement Fig. 12(a) and (b), while the hyperpiano was used to implement Fig. 12(c). The purpose of this example is to impose a traditional common-practice tonal paradigm onto the hyperoctave system in order to test for functional compatibility between the two systems. Informal listening suggests that cadence (b)

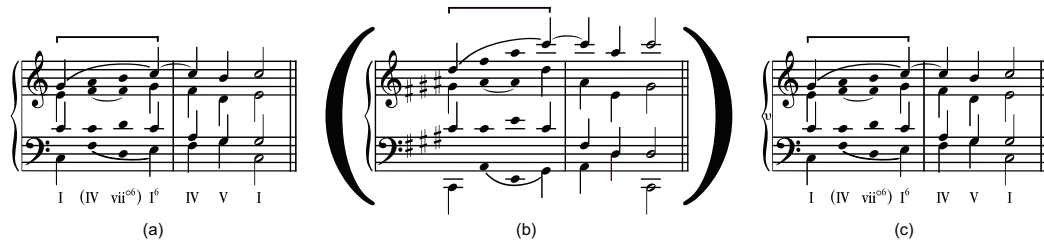


Figure 12: A conventional common-practice harmonic cadence on a grand staff (a) is transposed into a hypercadence on a grand staff (b), and then that same hypercadence is depicted on a grand hyperstaff (c).

should sound harsh and disjointed compared with cadences (a) and (c). More study is required to properly ascertain the robustness of the tonal potential inherent to the hyperoctave system. For example, the extra-wide leading tone in (c) may cause it to sound less final than the smaller and more familiar leading tone in (a).

7 Conclusions and future work

This paper documents the design and construction of the hyperpiano, a musical instrument that showcases the sound of a nonuniform string that produces an inharmonic spectrum. By using readily available materials such as a second hand piano, commercially available wound strings that can be partially “unwound” to create the nonuniform string contour, and by carefully simulating each part of the process (both the materials and the sound), the cost was kept well below that of a new mass-produced instrument. Many decisions were required throughout the design procedure. Other choices would have led to instruments using different proportions of wound to unwound string, to different sound spectra, to different tunings, and to different keyboard layouts. We selected the hyperpiano 12-het system partly because of its relationship with both 12-tet and with the whole tone scale; in this way it provides a unique blend of the familiar and the unfamiliar.

The hyperoctave system itself is not closed; one way to extend it would be to exploit the identity of the intervals in the $\sqrt[12]{2}$ chromatic scale and the quarter-hypertone scale with steps $\sqrt[24]{4}$. Well established quarter-tone notation could be used and a two-manual piano could be constructed (similar to the quarter-tone piano developed by August Förster in 1923 [26]) on which quarter-hypertone music could be played.

More generally, there are other possible musical instruments based on inharmonic strings. For example, in the hyperpiano design, all strings use the same ratio of wound

to unwound segments. Clearly, it should be possible to use a collection of different ratios. The advantage would be a greater variety of tone colors; the disadvantage would be the greater complexity of stringing. Another area for future work would be to build a fretted instrument that uses nonuniform strings. Each fret would change both the length of the string (changing the pitch, as usual) as well as the relative lengths of the wound to unwound segments (hence changing the timbre). The advantage over a harp-based instrument would be that each string could produce several notes, each with its own timbre; the disadvantage would be the difficulty of designing and stringing the instrument. Finally, another challenge would be to design other vibrating elements with compatible inharmonic spectra. For example, it is likely that bells can be designed to match the first five partials of the hyperpiano string spectrum, as McLachlan shows in [27]. Such investigations could lead to the creation of complete “orchestras” of acoustic instruments compatible with the hyperpiano or other given designs.

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