An Approach to Blind Equalization of Non-Minimum Phase Systems **D1.6**

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ABSTRACT: Blind equalization of systems which contain a nonminimum phase component is a notoriously difficult task. Minimizing the energy (l_2 minimization) of the equalizer output (under a fixed tap constraint) cannot be guaranteed to open the eye (to reliably unscramble the message) because it tends to converge to an equalizer setting that contains a reflection of the unstable zeros inside the unit circle [1]. We show that in at least one simple setup involving a mixture of minimum and nonminimum phase elements, an l_∞ minimization of the equalizer output is the appropriate criterion which should be minimized in order to successfully open the eye. Utilizing a finite impulse response equalizer which is constrained to have a unity coefficient on the center tap, we show that, for large enough dimension (depending on the closeness of the zeros of the channel to the unit circle) the eye will be opened. Unfortunately, there is no simple (gradient) scheme to exactly implement the l_∞ minimization, and we propose using a gradient l_p scheme, for p large. We show that there is generically a unique global equilibrium to which the scheme converges under weak excitation conditions on the input data. This result is in sharp contrast to competing schemes such as the Sato algorithm [2], the Godard algorithms [3] and the Constant Modulus Algorithms [4, 5] which are all susceptible to undesirable local equilibria [6].

1 Introduction

Communication channels invariably distort the data that they carry. The goal of channel equalization is to undo the effects of the distortion by building an equalizer that can be thought of as an inverse to the channel. This task is more difficult than standard system identification because the input is unavailable for use in the construction of the inverse, though it is made easier due to the nature of the data sent, which can often be modelled as a binary (or slightly more complex, as a M-ary or QAM) signal.

Figure 1 shows the basic setup of the channel and equalizer. Suppose that the input u_k consists of a binary signal that takes on the values ± 1, that the channel is represented by a (possibly nonminimum phase) finite impulse response (FIR) filter ϕ, and that the task of the FIR equalizer ξ is to make the reconstructed signal δ_k equal to the input, though possibly delayed in time. This goal is often refered to as "opening the eye (diagram)" of the channel, and may be stated as a requirement that the impulse response of

$$\gamma = \phi * \xi \tag{1.1}$$

(the convolution of ϕ and ξ) contains a single tap that dominates all the rest. Several remarks are in order regarding this choice of problem setup in which our goal is to highlight the features unique to nonminimum phase equalization. The channel is assumed FIR because the presence of minimum and nonminimum phase zeros is the most difficult part of the blind equalization task. The input is assumed binary, though extensions to the more useful QAM setup appear to be viable (the PAM generalization is immediate). The equalizer itself is chosen to be FIR because of the stability problems inherent in identification of autoregressive components (problems which are compounded by the nonminimum phase character of the channel).

Several algorithms have been proposed to carry out the identification of appropriate equalizer parameters. These typically involve minimizing the mean square energy of the equalizer output signal under a constraint on the equalizer parameters to avoid a degenerate solution, e.g, an off-line procedure is described in [1], or of minimizing the deviation of the equalizer output from a desired constant (modulus) value [2, 3, 4, 5]. In contrast, this paper shows that these are inappropriate things to minimize, and demonstrates that a more suitable criterion, in light of the goal of "opening the eye", is to minimize the l_∞ norm of the equalizer output δ (as opposed to the l_2 norm) under a fixed center tap constraint (as opposed to fixing a leading tap). This theory was generated independently to closely related work of Rupprecht [7, 8, 9]. Moreover, we propose a simple recursive algorithm that suitably approximates the desired l_∞ solution. The l_∞ approach generically has a unique global minimum which tends to open the eye (provided that the equalizer contains enough taps for an open eye solution to exist).

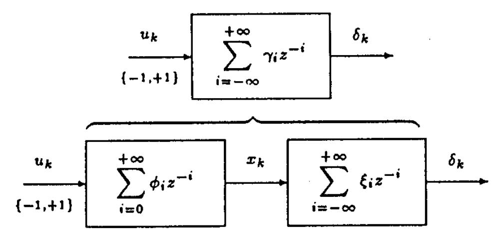

Figure 1: Channel and Equalizer

2 Problem Formulation and Notation

Throughout this paper, we focus on the task of constructing an equalizer for the FIR system (channel)

$$x_k = u_k + \phi_1 u_{k-1} + \phi_2 u_{k-2} \tag{2.1}$$

with transfer function (letting $\phi_0 = 1$, $\phi_1 = -a-b$, $\phi_2 = ab$, and $\phi_i = 0$ otherwise)

$$(1 - az^{-1})(1 - bz^{-1}) \tag{2.2}$$

where one zero is assumed inside the unit circle and one outside, $|b| > 1 > |a| \geq 0$. (For values of a and b not satisfying this condition we have either a minimum or maximum phase system for which successful blind algorithms are known to exist, using simple energy criteria like that described in [1].) Just as any AR filter can be approximated arbitrarily closely by an appropriate (doubly infinite) FIR filter, so too can any FIR filter with zeros not on the unit circle be approximated in (perhaps doubly infinite) AR form. Corresponding to (2.1) and (2.2) is the inverse of

$$\frac{z}{a-b}(\cdots + b^{-2}z^2 + b^{-1}z^1 + 1 + az^{-1} + a^2z^{-2} + \cdots)$$

which can be written

$$\frac{(a-b)z^{-1}}{\sum_{i=-\infty}^{\infty} \Xi_i z^{-i}} \tag{2.3}$$

where

$$\Xi_i = \begin{cases} a^i & i \geq 0 \\ b^i & i \leq 0 \end{cases} .$$

Note that the numerator factor in (2.3) is inconsequential to the identification problem in the telecommunication context because it does not affect the relative eye opening. Further, observe that a purely minimum phase channel would tend to have $\Xi_i = 0$ for all positive i, while a purely maximum phase channel would tend to have $\Xi_i = 0$ for negative i. This describes what an ideal equalizer would be, since with $\Xi_i = \xi_i$ for all i, the convolution $\gamma = \phi * \xi$ contains a single non-zero coefficient. Thus

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an equalizer constructed in this fashion fully opens the eye. In practice, the channel must be truncated to a finite length, and the noncausality can be handled by delaying the finite equalizer an appropriate amount [10]. Nevertheless, this provides a benchmark against which the success or failure of a candidate blind equalization scheme can be measured.

3 Comparison of Blind Equalization Schemes

It is worthwhile to compare the various blind equalization schemes in terms of their ability to correctly open the eye for the benchmark system of the previous section. There are two questions which must be asked. First, is the true equalizer a solution point of the candidate blind equalization scheme? Second, is this the only solution? The first subsection demonstrates that the minimum energy methods are unimodal, but that the point of minimum energy may not be the best equalizer in the sense that it may fail to open the eye even when an open eye solution exists. The second subsection shows that the minimum variance methods do always contain the correct equalizer as a solution, but that there are also other, local minima to which the scheme may converge. Finally, we introduce the criteria of minimizing the l_∞ norm of the output of the equalizer. This error criteria may contain "the best of both worlds", in that the true equalizer is a solution, and that this solution is, in fact, globally attractive (in the sense that the error criterion as a function of the equalizer tap parameter values is convex and hence uni-modal).

3.1 Minimum Energy Methods

One way to approach the blind equalization problem is to set up a "cost" function that is the mean squared error of the output of the channel-equalizer pair, and to attempt to minimize the cost by choice of the equalizer parameters. That is, one seeks to find the minimum of

$$E\left\{|\delta_j|^2\right\} = \left\|\gamma\right\|_2^2 = \left\|\Phi\xi\right\|_2^2 \qquad (3.1)$$

where

$$\Phi \triangleq \begin{pmatrix} \ddots & & & & & \\ & 1 & & & & \\ & \phi_1 & 1 & & & \\ & \phi_2 & \phi_1 & 1 & & \\ & & \phi_2 & \phi_1 & & \\ & & & \phi_2 & & \\ & & & & & \ddots \end{pmatrix}$$

is a channel convolution matrix, and

$$\xi \triangleq (\cdots, \xi_{-2}, \xi_{-1}, \xi_0, \xi_1, \xi_2, \cdots)^T \qquad (3.2)$$

is the equalizer. (In (3.1) we have assumed an i.i.d., property on the input symbols.) If applied naively, this approach is useless, because the minimum energy solution is clearly at $\xi = 0$. One solution [11] (based on related material [1]) and [7, 8, 9], is to fix one of the taps, say

$$\xi_0 = 1. \qquad (3.3)$$

Then the minimization has a nontrivial answer. So $\|\Phi\xi\|_2$ can be rewritten by excising the column corresponding to the term ξ_0 (and labelling it as y) as

$$\left\| \begin{pmatrix} \ddots & & & & & \\ 1 & & & & & \\ \phi_1 & 1 & & & & \\ \phi_2 & \phi_1 & & & & \\ & \phi_2 & 1 & & & \\ & & \phi_1 & 1 & & \\ & & \phi_2 & \phi_1 & & \\ & & & \phi_2 & & \\ & & & & & \ddots \end{pmatrix} \begin{pmatrix} \vdots \\ \vdots \\ \xi_{-2} \\ \xi_{-1} \\ \xi_1 \\ \xi_2 \\ \vdots \\ \vdots \end{pmatrix} + \begin{pmatrix} \vdots \\ 0 \\ 0 \\ 1 \\ \phi_1 \\ \phi_2 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \right\|_2 = \left\|\Phi_{(d)}\xi_{(d)} + y\right\|_2$$

where the subscript (d) indicates the removal of the appropriate column from Φ and the corresponding element of ξ. When the equalizer has finite dimension, the solution to this minimization problem can be written in closed form using the pseudo-inverse

$$\xi_{(d)} = -(\Phi_{(d)}^T\Phi_{(d)})^{-1}\Phi_{(d)}^T y.$$

The infinite dimensional version of this falters because the matrix $\Phi_{(d)}^T\Phi_{(d)}$ fails to be invertible, a fact which was exploited in [11].

For example, consider the particular channel a=0.5, b=3.0 ($\phi_1 = -3.5$, $\phi_2 = 1.5$). With 10 adjustable taps in the equalizer (i.e., $n = 11$), the above equations can be solved readily to give

$$\xi = (0.041, 0.112, 0.230, 0.424, 0.713, 1.000, 0.713, 0.424, 0.230, 0.112, 0.041)^T.$$

Despite the fact that $a \neq b^{-1}$, (recall that the optimum equalizer is defined by (2.3)) this minimum energy solution is symmetric. In fact, this ξ fails to even open the eye, as can be readily calculated by noting that the convolution of ϕ and ξ is not dominated by a single tap (the largest element of γ is 1.72, and the sum of the other elements is 4.17). Thus, despite the fact that the minimum energy solution is unique, it is not necessarily the right criterion to minimize. (For minimum phase systems the minimum energy solution is satisfactory [1].) This symmetry of the answer is maintained by the minimum energy answer no matter how large an equalizer is chosen. For such nonminimum phase channels, then, there is no reason to expect the minimum energy solution to give a viable equalizer in general (see [10] for related issues).

3.2 Constant Modulus Algorithms

Algorithms of this class were developed in [2, 3, 10, 4, 5]; from which we select the so-called constant modulus algorithm (CMA) (actually first developed in [3]) as a representative example. The idea behind CMA is that it tries to minimize the variance or deviation of the channel output from its desired value of ± 1 (since the symbols are binary). The criterion is to minimize

$$E\left\{(|\delta_j|^2 - 1)^2\right\}$$

by choice of ξ. Clearly, an equalizer which forms the exact inverse (modulo a scaling and a time delay), e.g., like the one in section 2, fulfills this criterion, and it can be shown that this answer is a stable equilibrium [12]. Unfortunately, there is also the possibility that the equilibrium is not unique, that there may exist other, local minima to which the algorithm may converge. This is indeed the case, assuming that the number of taps is finite. For instance, in [6] it is shown that if the channel has a single stable pole with IIR transfer function $\sum_{i=0}^{\infty} \alpha^i z^{-i}$, then there exist undesirable local minima for a two tap equalizer. (Note two taps suffice to form the exact inverse of such a channel.) Examples such as these easily extend to the case where an arbitrary but finite number of taps are utilized. Thus there are undesirable local minima corresponding to equalizer parameter values which do not even approximate the channel inverse. If the number of taps in the equalizer is allowed to be infinite then no local minimum can occur [13, 10], though there may be numerous saddle points. This infinite tap case is primarily of theoretical interest.

3.3 Minimum l_∞ of Equalizer Output

The failure modes of the above algorithms indicate the need for a better approach. Indeed, consider the problem from the point of view of opening the eye. In terms of the channel-equalizer convolution γ, the eye is open when $|\gamma_{max}|$ dominates $\sum_{i\neq max}|\gamma_i|$. Obviously, this sum is closely related to an l_1 (sum of absolute values) criterion. In terms of norms as in section 3.1, this l_1 tap minimization is equivalent to a l_∞ (also known as sup norm) minimization of the output δ_k [14] of the equalizer assuming the input is sufficently rich but not necessarily i.i.d. This line of reasoning suggests that it might be advantageous to minimize

$$\left\|\delta\right\|_\infty \triangleq \max_{\{u_k\}}|\delta_k| = \left\|\gamma\right\|_1 = \left\|\Phi\xi\right\|_1 = \left\|\Phi_{(d)}\xi_{(d)} + y\right\|_1 \qquad (3.4)$$

with notation as in section 3.1. (A similar idea appears (independently) in [7, 8] and in a more refined form in [9].) Again, it is essential to insist that at least one element of ξ be nonzero, as in (3.3), to avoid the trivial $\xi = 0$ solution.

In the next section we demonstrate that the error surface associated with (3.4) is always convex with respect to the adapted taps $\xi_{(d)}$ (this result is also proven in [9]), and that the resulting minimization will always open the eye, assuming the equalizer is of sufficient length, for any channel of the form (2.1). While simulations indicate that the method works for many more complex systems, we hesitate to claim without proof that it must open the eye for all possible channels (except those with zeros on the unit circle). In fact, later we indicate that at least for a non-generic class of channels there exists a problem with non-uniqueness of solution to the l_∞ minimization.

This section demonstrates that the l_∞ minimization gives the correct desired equalizer in the cases above for which the minimum energy and minimum variance schemes fail. Using the standard l_1 minimization package from the NAG library [15, 16], it is easy to show that for the channel with a=0.5 and b=3.0, and with ten adjustable taps ($n = 11$) subject to (3.3), the l_∞ output criterion gives

$$\xi = (0.003, 0.012, 0.037, 0.111, 0.333, 1.000, 0.500, 0.250, 0.124, 0.060, 0.026)^T.$$

which is very nearly an exact truncation of the ideal value (2.3).

For the channel in section 3.2 which caused CMA to misbehave we find again that the l_∞ minimization gives the desired answer (modulo truncation). For example, if $\alpha = 0.7$ (pole position) and $n = 11$, then the l_∞ minimization gives

$$\xi = (0.0, 0.0, 0.0, 0.0, 0.0, 1.000, -0.700, 0.0, 0.0, 0.0, 0.0)^T.$$

4 Theoretical Development

This section presents two results. First, the error surface over which the minimization is performed (i.e., subsequently upon which the adaptive algorithm performs a gradient descent) is shown to be convex. This implies that any local minimum must also be a global minimum. Second, for channels of the type (2.1), the desired equalizer (2.3) is shown to be a solution to the minimization of the norm (3.4).

Theorem 1: $\|\gamma\|_1 \triangleq \sum_i |\gamma_i|$ is convex with respect to ξ.

Proof: We have

$$\|\gamma\|_1 = \sum_i |\gamma_i| = \sum_i \Big|\sum_j \phi_j \xi_{i-j}\Big|$$

noting the γ weights are a linear function of the ξ equalizer taps; so write $\gamma \equiv \gamma\{\xi\}$. Let $\xi^a \in l_1$, $\xi^b \in l_1$, and $0 \le \eta \le 1$. Then

$$\begin{aligned}\Big\|\gamma\big[(1-\eta)\xi^a + \eta\xi^b\big]\Big\|_1 &= \sum_i \Big|\sum_j \phi_j\big((1-\eta)\xi^a_{i-j} + \eta\xi^b_{i-j}\big)\Big| \\ &\le \sum_i \Big|\sum_j \phi_j(1-\eta)\xi^a_{i-j}\Big| + \sum_i\Big|\sum_j \phi_j\eta\xi^b_{i-j}\Big| \\ &= (1-\eta)\big\|\gamma[\xi^a]\big\|_1 + \eta\big\|\gamma[\xi^b]\big\|_1.\end{aligned}$$

□

Two remarks are in order. Firstly, this convexity is unaffected by the number of equalizer taps, be they finite in number, semi-infinite or doubly infinite. Secondly, fixing a center tap (or applying any linear constraint for that matter) does not destroy convexity.

From the above we can conclude that fixing the central equalizer tap at $\xi_0 = 1$ implies that $\|\delta\|_\infty$ is convex with respect to $\xi_{(d)}$, see (3.4). Further, and significantly, any truncation or finite equalizer approximation necessary in practice will not destroy this convexity.

Next, we examine the solution to the l_∞ problem by defining the function

$$f(\xi) \triangleq \sum_j |\gamma_j| = \sum_j \Big|\sum_i \phi_{j-i}\xi_i\Big| \tag{4.1}$$

and showing that at the desired equalizer setting Ξ (as defined in (2.3)), the directional derivatives are all nonnegative. This demonstrates that the desired answer is a solution to the l_∞ minimization problem, at least for the simple class of channels (2.1).

Theorem 2: Consider a 2 tap FIR channel with impulse response (2.1). Let Ξ denote the (doubly infinite) FIR inverse of (2.1) as in (2.3). Then, with the constraint that $\xi_0 = 1$, $\xi_i = \Xi_i$ minimizes $\big\|\Phi\xi\big\|_\infty$.

Proof: Pick a nonzero bounded vector Δ with $\Delta_0 = 0$. We examine the directional derivate of $f(\cdot)$, and show that it is always nonnegative. Note that

$$f(\xi + h\Delta) = \sum_j \Big|\sum_i \phi_{j-i}(\xi_i + h\Delta_i)\Big| \tag{4.2}$$

Let $v_j = \sum_i \phi_{j-i}\Delta_i$. Then

$$f(\xi + h\Delta) = \sum_j \big|\gamma_j - hv_j\big| \tag{4.3}$$

where γ is the convolution of the channel (2.1) and the equalizer ξ. Since $\xi = \Xi$ is the inverse of (2.1), $\gamma_0 = 1$ and $\gamma_i = 0$ for $i \ne 0$. Breaking the sum into two pieces corresponding to where $\gamma_i = 0$ and $\gamma_i \ne 0$ gives

$$= |1 + hv_0| + h\sum_{j\ne 0}|v_j|. \tag{4.4}$$

For h small (such that $|hv_0| < 1$),

$$1 - |hv_0| \le |1 + hv_0| \le 1 + |hv_0| \tag{4.5}$$

which implies that

$$1 + h\sum_{j\ne 0}|v_j| - |h||v_0| \le f(\xi + h\Delta) \le 1 + h\sum_{j\ne 0}|v_j| + |h||v_0|. \tag{4.6}$$

The right hand derivative in the direction Δ

$$f'_\Delta = \lim_{h\to 0+}\frac{f(\xi + h\Delta) - f(\xi)}{h} \tag{4.7}$$

can be bounded

$$\sum_{j\ne 0}|v_j| - |v_0| \le f'_\Delta \le \sum_{j\ne 0}|v_j| + |v_0|. \tag{4.8}$$

since $f(\xi) = \sum_j |\gamma_j| = 1$. For the simple channel with one zero inside the unit circle and one zero outside, it is easy to check that $|v_0| \le \sum_{j\ne 0}|v_j|$ due to the requirement that $\Delta_0 = 0$. The left hand derivate can be calculated exactly as above, to show that all derivatives are nonnegative. □

Theorem 2 supposes that an infinite length equalizer is available. Good approximations result, however, when the equalizer is truncated, and simulations indicate that the length need not be excessive. Note that the required length of the equalizer is not a limit of the l_∞ approach, but rather is a fundamental limit on the existence of an equalizer for the given channel.

5 Algorithm Development

5.1 l_p Norm Approximation

The results of the previous section would be of a purely academic interest if it were not possible to design a simple and effective recursive adaptive scheme to implement the minimization. In fact, general l_1 minimization is complicated algebraically and is equivalent to a linear programming problem [15]. Exact implementation of the algorithm is therefore precluded in a real time setting. However, due to the structure of the problem, a simple and effective way to approximate the desired minimization arbitrarily closely is available. Assuming a finitely parametrized equalizer (recall that l_∞ convexity is preserved when truncating from infinite to finite parameterizations), denote the finite equalizer (version of (3.2)) by

$$\hat\xi \triangleq (\hat\xi_{-n}, \cdots, \hat\xi_{-1}, \hat\xi_0, \hat\xi_1, \cdots, \hat\xi_n)^T \tag{5.1}$$

which is subject to the fixed tap constraint (3.3).

Instead of minimizing the infinity norm of the output of the equalizer, consider minimizing the l_p norm of the output. (In the next section we verify that convexity is not destroyed by such an approximation.) For large p, the two problems will have answers which are virtually indistinguishable. Though there are potential numerical problems (for very large p), it appears that these can be overcome by a judicious normalization. The goal then, is to minimize $E\{\|\delta_k\|_p\}$, which can be accomplished via a gradient descent strategy.

Define the instantaneous cost $J \triangleq |\delta_k|^p$. Then the algorithm becomes

$$\widehat{\xi}_{(d)}(k+1) = \widehat{\xi}_{(d)}(k) - \mu \frac{\partial J}{\partial \widehat{\xi}_{(d)}(k)} = \widehat{\xi}_{(d)}(k) - \mu \frac{\partial |\delta_k|^p}{\partial \widehat{\xi}_{(d)}(k)}$$

where $\widehat{\xi}_{(d)}(k)$ is the vector of current estimates of the equalizer parameters excluding the $\widehat{\xi}_0$ parameter frozen at unity, δ_k is the current output of the equalizer, and μ is a small positive stepsize. To calculate the gradient term, recall (Fig.1) that the equalizer output may be written

$$\begin{aligned} \delta_k &= \sum_{i=0}^{2n} \widehat{\xi}_{i-n}(k) x_{k-i} \qquad (5.2) \\ &= \widehat{\xi}^T(k) X(k) = \widehat{\xi}^T_{(d)}(k) X_{(d)}(k) + x_{k-n} \end{aligned}$$

where $X(k) \triangleq (x_k, x_{k-1}, \cdots, x_{k-2n})^T$ is a regressor of past inputs to the equalizer. The (d) subscripted versions as before denote the excised versions. The gradient is then $(p \geq 2)$

$$\frac{\partial |\delta_k|^p}{\partial \widehat{\xi}_{(d)}(k)} = p \delta_k |\delta_k|^{p-2} X_{(d)}(k) \qquad (5.3)$$

which then gives the algorithm in its implementable form

$$\widehat{\xi}_{(d)}(k+1) = \widehat{\xi}_{(d)}(k) - \mu\, \delta_k \left|\frac{\delta_k}{\eta}\right|^{p-2} X_{(d)}(k) \qquad (5.4)$$

where the constant p has been absorbed into the stepsize μ and the factor η has been added as a normalization to help combat potential numerical problems when raising numbers to a high power. Naturally, the algorithm works.

5.2 Convexity Preservation Under Approximation

Theorem 1 addressed the convexity of the ideal l_∞ cost applied to the equalizer output δ_k (this is equivalent to the l_1-norm of the overall convolution of channel and equalizer (1.1)). However our algorithm used the l_p approximation and we need to be concerned whether we have detroyed the convexity. Fortunately, the answer is no.

Theorem 3: $E\{|\delta_k|^p\}$ is convex with respect to ξ.

Proof: We have

$$\frac{\partial}{\partial \xi} E\{|\delta_k|^p\} = E\{p \delta_k |\delta_k|^{p-2} X(k)\}.$$

Then the Hessian is given by

$$\frac{\partial}{\partial \xi^T}\left\{\frac{\partial}{\partial \xi} E\{|\delta_k|^p\}\right\} = E\{\underbrace{p(p-1)|\delta_k|^{p-2}}_{\text{scalar}} X(k) X^T(k)\} \geq 0$$

□

Note the above proof is not easily extended to demonstrate *strict* convexity since positive definiteness of the Hessian is not a necessary condition for strict convexity. Also, as before, convexity of $E\{|\delta_k|^p\}$ with respect to $\xi_{(d)}$ is guaranteed.

6 Comments

This paper has shown that for a class of channels blind equalization is possible using a convex l_∞ cost on the equalizer output. However, for a class of non-generic channels this approach can fail due to non-uniqueness of the cost minimization since the criterion is not *strictly* convex. To make this clearer, consider the following construction. Suppose a channel has an inverse with more than one maximal impulse response value. Then there exist at least two equalizer parameter settings with the same (hypothetically minimizing) cost and it is easy to check that convex combinations of these settings also achieve the same cost. Such convex combinations will not generically open the eye diagram.

For that non-generic class of channels one can cure the non-uniqueness of the global minimum by generalizing the fixed tap constraint to more general sets of linear constraints on the adapted taps (each constrained adaptation giving a different set of non-generic problem channels). This mathematical generalization is straightforward.

Implicit in our scheme is the need for automatic gain control (AGC) applied to the signal δ_k. This is particularly appropriate for the PAM generalization of our binary analysis, for which the algorithms we have developed apply without change. AGC corresponds to a simple scalar version of any of the blind algorithms developed in [2, 3, 5].

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