Stability of Active Noise Control Algorithms

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***Abstract*—This paper provides a stability analysis of a class of acoustic noise control algorithms by showing that the adapted models have more in common with (nonlinear, finite impulse response [FIR]) equation error models than with the infinite impulse response (IIR) output error models they superficially resemble. Stability results from the adaptive control literature are applied to show global stability in the noise free case, and to show exponential stability when the input is persistently excited. The latter demonstrates a robustness to mismodeling errors, disturbances such as noises, and allows results to be applied to the tracking of time-varying systems.**

***Index Terms*—Acoustic noise, adaptive control, exponential stability, parameter estimation.**

I. Introduction

THIS note presents a stability analysis of a class of algorithms used in acoustic noise control. Adaptive control techniques which concentrate on studying stability issues in systems with feedback and adaptation will be shown to be relevant to acoustic noise control problems due to the presence of acoustic feedback loops. Application areas include HVAC (heating, ventilation, and air conditioning) noise control and enclosure noise control for automobiles, aircraft, and elevators. The preferred compensator structure for parameter adaptive, feedforward, active noise control in ducts is a direct form infinite impulse response (IIR) model. Thus, adaptation algorithms and their analytical justification and interpretation have been sought among those intended for adaptive IIR filters in a parallel system identification configuration. Indeed, [1, p. 200] writes

> The properties of such adaptive algorithms for IIR filters are still not fully understood, especially when complicated by the physical feedback path present in this case.

The discussion of and modifications recommended in [2], [3] suggest an analysis concentrating on the IIR structure of the filter to be adapted. The point of this paper is to show that the IIR structure may be misleading and that the stability analysis can proceed by rewriting the problem in a nonlinear equation error formulation. This allows direct application of previous theoretical results from the adaptive control literature.

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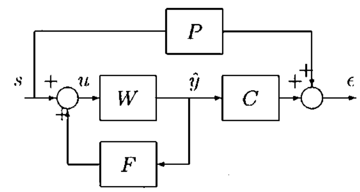

Fig. 1. Duct block diagram description.

II. A Stability Perspective on Algorithm Development

Active noise cancellation schemes such as those from [2], [4] can be portrayed as in Fig. 1. The following assumptions specify the problem setting in terms of the variables defined in Fig. 1.

A1) *Disturbance*. The source signal s is uniformly bounded

$$|s(k)| \leq \alpha \qquad \forall\, k. \tag{1}$$

A2) *Feedback*. The detection microphone signal u is described by

$$u(k) = s(k) + \sum_{i=1}^{Q_2} f_i \hat{y}(k-i) \tag{2}$$

where the coefficients f_i are bounded

$$|f_i| \leq \beta \qquad \forall\, i. \tag{3}$$

A3) *Error Signal*. The source propagated to the error microphone plus $\hat{y}$ yields (for $C = 1$ the error signal

$$\epsilon(k) = \hat{y}(k) + \sum_{i=0}^{Q_1} p_i s(k-i) \tag{4}$$

where the coefficients are bounded

$$|p_i| \leq \gamma. \tag{5}$$

A4) *Control Output*. The output of the adaptive filter is

$$\hat{y}(k) = \sum_{i=1}^{Q_3} \hat{a}_i(k-1)\hat{y}(k-i) + \sum_{i=0}^{Q_1} \hat{b}_i(k-1)u(k-i). \tag{6}$$

When $C = 1$, the order of the adaptive filter is assumed to match the order of the error signal (for the moving average portion) and the order of the feedback path plus the error signal (for the autoregressive portion), that is, $Q_3 = Q_1 + Q_2$. When C is order q, then $Q_3 = \max(Q_1 + Q_2, q)$.

A5) *Adaptive Algorithm*. The algorithm for updating the filter coefficients in (6) is based on the recursive least mean square (LMS) algorithm ([1, equation (6.15.5)]), modified by a "normalized" stepsize, which is common practice in both signal processing and control applications. The parameter adaptation algorithm is

$$\hat{a}_i(k) = \hat{a}_i(k-1) - \mu(k)\epsilon(k)\hat{y}(k-i) \tag{7}$$
$$\hat{b}_i(k) = \hat{b}_i(k-1) - \mu(k)\epsilon(k)u(k-i) \tag{8}$$

where the time-varying, normalized stepsize is

$$\mu(k) = \frac{\mu}{\delta + \sum_{i=1}^{Q_3} \hat{y}^2(k-i) + \sum_{i=0}^{Q_1} u^2(k-i)} \tag{9}$$

for small fixed positive values of δ and μ.

The problem set up A1–A5 captures a fairly typical, if somewhat simplified adaptive noise cancellation setting. As shown below, this can be rewritten as an "adaptive control" problem in which the adaptation of the $\hat{a}_i$s and $\hat{b}_i$s are viewed as an implicit identification of an unknown system B/A. For instance, when $C = 1$, $W = (-P/(1-FP))$ causes the error signal ϵ to be identically zero and hence, $B \leftrightarrow -P$ and $A \leftrightarrow 1 - FP$ define the system to which the $\hat{a}_i$ and $\hat{b}_i$ must converge in order to minimize the squared error.

Accordingly, define the "unknown" parameter vector

$$\theta_0 = [a_1, \cdots, a_{Q_3}, b_1, \cdots, b_{Q_1}]^T \tag{10}$$

(for the $C = 1$ case, $b_i = -p_i$ and $a_\ell = \sum_{i+j=\ell} p_i f_j$). The "parameter estimates" are defined to be

$$\hat{\theta}(k) = [\hat{a}_1(k), \cdots, \hat{a}_{Q_3}(k), \hat{b}_0(k), \cdots, \hat{b}_{Q_1}(k)]^T$$

and the parameter error vector is $\tilde{\theta}(k) = \hat{\theta}(k) - \theta_0$. Also, define the "regressor vector"

$$\phi(k-1) = [\hat{y}(k-1), \cdots, \hat{y}(k-Q_3), u(k), \cdots, u(k-Q_1)]^T$$

and the prediction error

$$e(k) = \phi(k-1)^T \tilde{\theta}(k-1). \tag{11}$$

The "projection algorithm" from Goodwin and Sin [5] is then

$$\hat{\theta}(k) = \hat{\theta}(k-1) - \frac{\mu\phi(k-1)}{\delta + \phi(k-1)^T\phi(k-1)} e(k). \tag{12}$$

Theorem 2.1: Consider the acoustic noise control problem as shown in Fig. 1 with $C = 1$ under assumptions A1–A5. Then, the following applies.

1) The error signal converges to zero for all initial conditions, that is,

$$\epsilon(k) \to 0 \quad \text{as} \quad k \to \infty.$$

2) The regressor $\phi(k)$ is uniformly bounded.

Proof: In order to apply [5, Lemma 3.3.2], it is only necessary to show that $\epsilon(k)$ of (4), (7) and (8) is equal to $e(k)$ of (11) and (12). Combining (6) and (4) gives

$$\epsilon(k) = \sum_{i=0}^{Q_1} p_i s(k-i) + \sum_{i=1}^{Q_3} \hat{a}_i(k-1)\hat{y}(k-i) + \sum_{i=0}^{Q_1} \hat{b}_i(k-1)u(k-i). \tag{13}$$

Equation (2) can be rewritten as

$$s(k-i) = u(k-i) - \sum_{j=1}^{Q_2} f_j \hat{y}(k-j-i)$$

which can be used in (13) to give

$$\epsilon(k) = \sum_{i=0}^{Q_1} p_i u(k-i) - \sum_{i=0}^{Q_1} p_i \sum_{j=1}^{Q_2} f_j \hat{y}(k-i-j) + \sum_{i=1}^{Q_3} \hat{a}_i(k-1)\hat{y}(k-i) + \sum_{i=0}^{Q_1} \hat{b}_i(k-1)u(k-i).$$

Since $a_\ell = \sum_{i+j=\ell} p_i f_j$ and $b_i = -p_i$, this can be rewritten

$$\begin{aligned}\epsilon(k) &= \sum_{i=1}^{Q_3} [\hat{a}_i(k-1) - a_i]\,\hat{y}(k-i) + \sum_{i=0}^{Q_1} \left[\hat{b}_i(k-1) - b_i\right] u(k-i) \\ &= \phi(k-1)^T \tilde{\theta}(k-1) = e(k).\end{aligned}$$

To apply the "key technical lemma" 6.2.1 of [5], observe that the bounds in (1) and (5) can be substituted into (4) to give

$$\hat{y}(k) \le |\hat{y}(k)| \le Q_1 \gamma \alpha + \max_{0 \le \tau \le k} |\epsilon(\tau)|. \tag{14}$$

Similarly, the bounds (3) and (5) are substituted into (2) to give

$$|u(k)| \le \gamma + Q_2 \beta \max_{0 \le \tau \le k} |\hat{y}(\tau)|. \tag{15}$$

Merging (14) and (15) shows that $\phi(k)^T\phi(k) \le C_1 + C_2 \max_{0 \le \tau \le k} |\epsilon(\tau)|$ and hence, [5, Lemma 6.2.1] provides the desired conclusions.

Remarks:

1) This stability result applies to "recursive LMS" algorithms based on a (nonlinear) equation error, rather than an output error, problem formulation. The nominal behavior is that of a globally stable algorithm.
2) The uniform boundedness of the regressor is with respect to any norm in $\mathbb{R}^{2Q_1+Q_2+1}$ over time.
3) Theorem 2.1 also shows that $\epsilon(k) \in \ell_2$, that is, the prediction error sequence is square summable.

This result can be generalized to the case when C is any strictly positive real (SPR) transfer function. To see this, redraw the block diagram, pulling C through the summation node on the right and replacing P with P/C to compensate. Then the error signal ϵ is precisely zero when $W = (-P/(C - FP))$ and hence, $B \leftrightarrow -P$ and $A \leftrightarrow C - FP$ define the system to which the $\hat{a}_i$ and $\hat{b}_i$ must converge in order to minimize the squared error. Accordingly, redefine Q_3 of (6) to be the maximum of the order of C and the order of the feedback path plus the error signal.

Corollary 2.1: Consider the acoustic noise control problem as shown in Fig. 1 under assumptions A1–A5 and with C SPR. Then the error signal converges to zero for all initial conditions, that is, $\epsilon(k) \to 0$ as $k \to \infty$, and the regressor $\phi(k)$ is uniformly bounded.

Proof: Follows exactly as in Theorem 2.1 but with a_ℓ redefined as $a_\ell = -c_\ell + \sum_{i+j=\ell} p_i f_j$. The use of $C\{\epsilon(k)\}$ instead of $\epsilon(k)$ as the error term in the algorithm does not effect the convergence and stability properties as long as C is SPR, as shown in the lemma proven in [6].

Remarks:

1) The case when C is not SPR remains an open issue.
2) Because of the equation error structure, regressor and/or error filtering such as that used in [7] are not needed in the adaptive algorithm.

The conclusions of Theorem 2.1 imply that $||\hat{\theta}(k) - \theta_0|| \leq ||\hat{\theta}(k-1) - \theta_0||$ for all k. Hence, once an estimate of a given quality is made, the algorithm never makes a worse estimate. However, the theorem does not imply that the parameter estimates $\hat{\theta}$ converge to their desired values θ_0. But if an assumption of richness or complexity of the regressor is made, then this convergence can be guaranteed. Moreover, because the convergence is exponential, it implies a certain robustness to noises or disturbances that are inevitably present in any implementation.

A6) *Persistence of Excitation*. Assume that the unknown system θ_0 represents a stable linear filter and that the regressor vector satisfies

$$\frac{1}{K} \sum_{k=\ell}^{\ell+K-1} \frac{\phi(k)\phi(k)^T}{1+\phi(k)^T\phi(k)} \geq \alpha I > 0 \tag{16}$$

for some $\alpha > 0$, some $K > 0$, and all ℓ.

Theorem 2.2: Consider the acoustic noise control problem as shown in Fig. 1 under assumptions A1)–A6) and with C SPR. Then the parameter estimates $\hat{\theta}(k)$ converge exponentially to θ_0 from any initial condition.

Proof: With $C = 1$, this is a straightforward application of [8, Th. 3.5.19]. With C SPR, [7, Appendix] guarantees that

$$\frac{1}{K} \sum_{k=\ell}^{\ell+K-1} \frac{\phi(k)C\{\phi(k)^T\}}{1+\phi(k)^T C\{\phi(k)\}} \geq \alpha_2 I > 0$$

whenever (16) holds. Combining these two gives the desired exponential stability.

Remarks:

7) The persistence of excitation condition on the regressor vector ϕ can be transferred to a similar condition on the signals u (and hence ultimately onto s) following the development in [9].
8) Exponential stability is a strong property. It guarantees that the algorithm (12) remains stable even under persistent perturbations, that is, it guarantees robustness to small noises or disturbances in the inputs, outputs, measurements, and modeling parameters that inevitably arise in implementation.
9) Theorem 2.2 also guarantees that if the underlying systems (the P, F, and C transfer functions) change slowly, then the algorithm will track the changes. (see [8, Th. 3.5.29] for a detailed statement). Hence, robustness to time variations is also guaranteed.

III. Conclusion

This note has presented a stability analysis of commonly used recursive algorithms for active acoustic noise control. The analysis exploits adaptive control techniques by showing that the algorithms have stability properties closer to those of a nonlinear equation error (the projection algorithm) than to the output error form often associated with IIR adaptive filtering [10].

References

[1] P. Nelson and S. Elliott, *Active Control of Sound*. London, U.K.: Academic, 1992.
[2] L. Eriksson, "Development of the filtered-u algorithm for active noise control," *J. Acoust. Soc. Amer.*, vol. 89, pp. 257–265, Jan. 1991.
[3] L. Eriksson, M. Allie, and R. Greiner, "The selection and application of an IIR adaptive filter for use in active sound attenuation," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-35, Apr. 1987.
[4] S. Kuo and D. Morgan, *Active Noise Control Systems*. New York: Wiley, 1996.
[5] G. Goodwin and K. Sin, *Adaptive Filtering, Prediction and Control*. Englewood Cliffs, NJ: Prentice-Hall, 1984.
[6] Y. H. Lin and K. S. Narendra, "A new error model for adaptive systems," *IEEE Trans. Automat. Contr.*, vol. AC-25, pp. 585–587, June 1980.
[7] W. Sethares, B. Anderson, and C. Johnson Jr., "Adaptive algorithms with filtered regressor and filtered error," *Math. Contr. Signals Syst.*, vol. 2, pp. 381–403, 1989.
[8] I. Mareels and J. Polderman, *Adaptive Systems: An Introduction*. Boston, MA: Birkhäuser, 1996.
[9] B. Anderson and C. R. Johnson, "Exponential convergence of adaptive identification and control algorithms," *Automatica*, vol. 18, pp. 1–13, 1982.
[10] C. R. Johnson Jr., *Lectures on Adaptive Parameter Estimation*. Englewood Cliffs, NJ: Prentice-Hall, 1988.